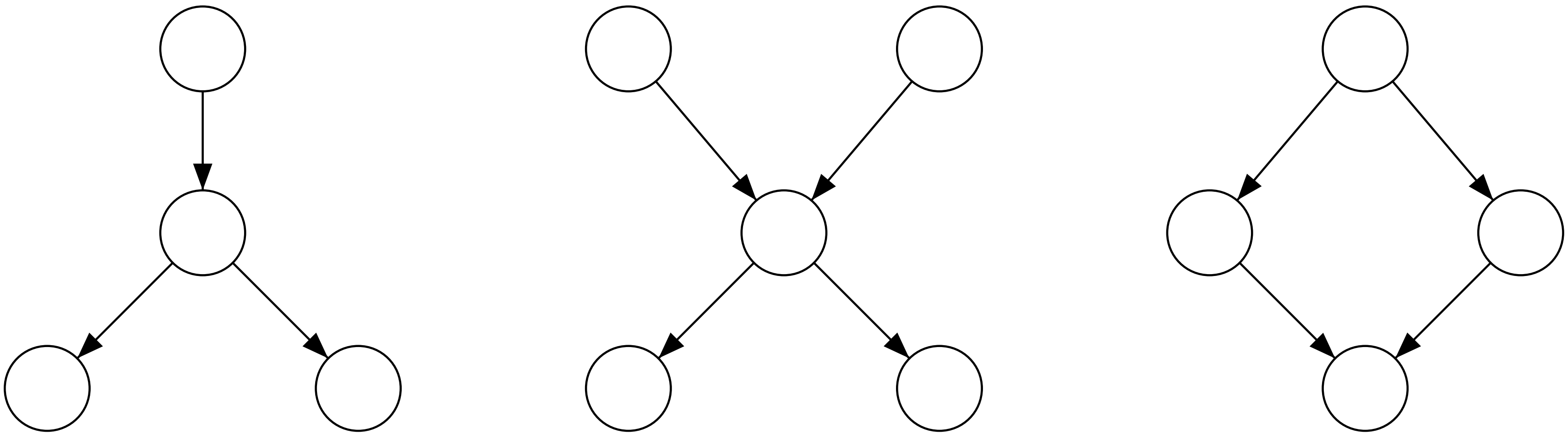
Clique Tree Representations

Problems



The propagation algorithm as presented can only deal with *trees*.

Can be extended to *polytrees* (i. e. singly connected graphs with multiple parents per node).

However, it cannot handle networks that contain loops!

Idea

Main Objectives:

Transform the cyclic directed graph into a secondary structure without cycles.

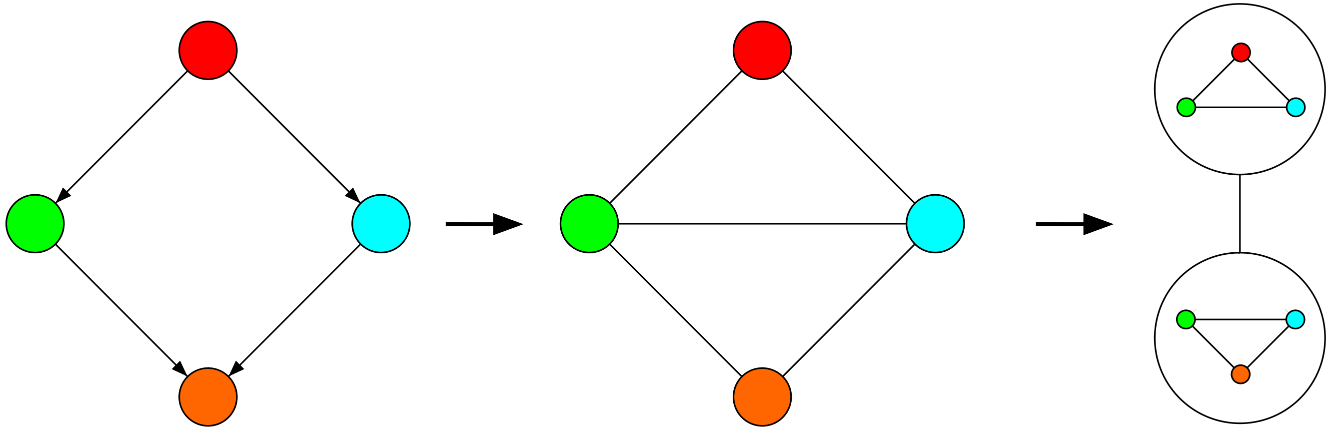
Find a decomposition of the underlying joint distribution.

Task:

Combine nodes of the original (primary) graph structure.

These groups form the nodes of a secondary structure.

Find a transformation that yields tree structure.



Secondary Structure:

We will generate an undirected graph mimicking (some of) the conditional independence statements of the cyclic directed graph.

Maximal cliques are identified and form the nodes of the secondary structure.

Specify a so-called potential function for every clique such that the product of all potentials yields the initial joint distribution.

In order to propagate evidence, create a **tree** from the clique nodes such that the following property is satisfied:

> If two cliques have some attributes in common, then these attributes have to be contained in every clique of the path connecting the two cliques. (called the **running intersection property**, **RIP**)

Justification:

Tree: Unique path of evidence propagation.

RIP: Update of an attribute reaches all cliques which contain it.

Prerequisites

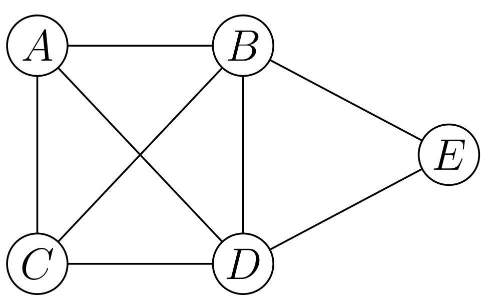
Complete Graph

An undirected Graph $G = (V, E)$ is called *complete*, if every pair of (distinct) nodes is connected by an edge.

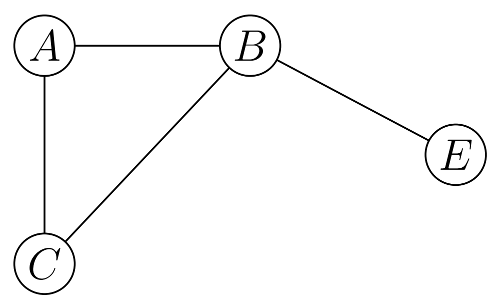
Induced Subgraph

Let $G = (V, E)$ be an undirected graph and $W \subseteq V$ a selection of nodes. Then, $G_W = (W, E_W)$ is called the *subgraph of G induced by W* with E_W being

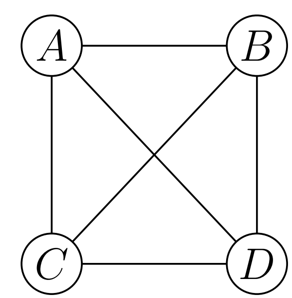
$$E_W = \{(u, v) \in E \mid u, v \in W\}.$$



Incomplete graph



Subgraph (W, E_W)
with $W = \{A, B, C, E\}$



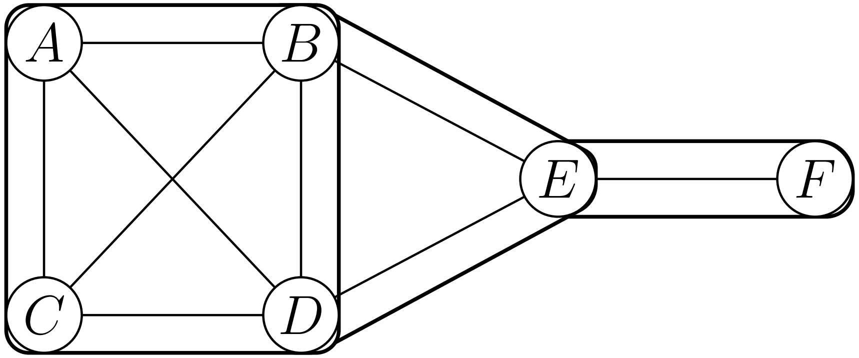
Complete (sub)graph

Complete Set, Clique

Let $G = (V, E)$ be an undirected graph. A set $W \subseteq V$ is called *complete* iff it induces a complete subgraph. It is further called a *clique*, iff W is maximal, i. e. it is not possible to add a node to W without violating the completeness condition.

a) W is complete $\Leftrightarrow$ W induces a complete subgraph

b) W is a clique $\Leftrightarrow$ W is complete and maximal



3 cliques

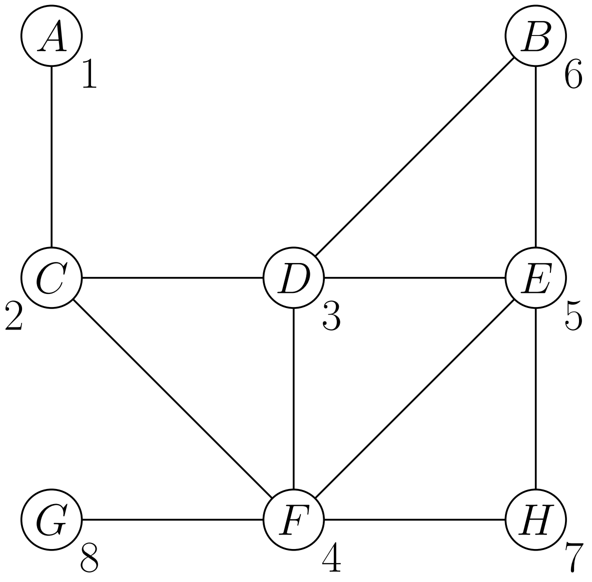
$$C_1 = \{A, B, C, D\}$$
$$C_2 = \{B, D, E\}$$
$$C_3 = \{E, F\}$$

Perfect Ordering

Let $G = (V, E)$ be an undirected graph with n nodes and $\alpha = \langle v_1, \dots, v_n \rangle$ a total ordering on V. Then, α is called *perfect*, if the following sets

$$\mathrm{adj}(v_i) \cap \{v_1, \dots, v_{i-1}\} \qquad i = 1, \dots, n$$

are complete, where $\mathrm{adj}(v_i) = \{w \mid (v_i, w) \in E\}$ returns the adjacent nodes of v_i.



$\alpha = \langle A,\ C,\ D,\ F,\ E,\ B,\ H,\ G \rangle$

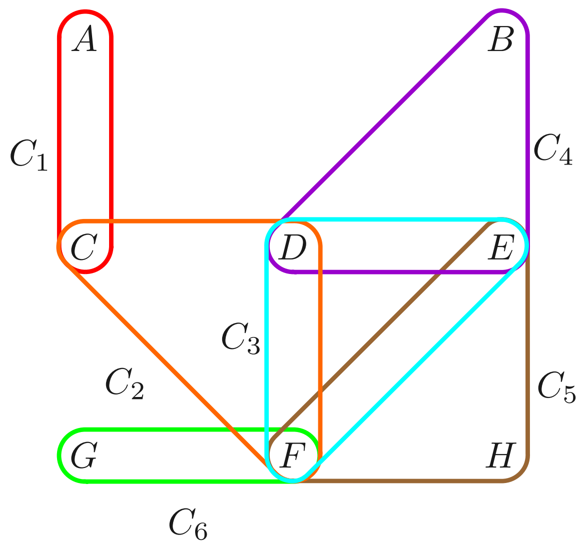
i	$\mathrm{adj}(v_i)$	$\{v_1, \dots, v_{i-1}\} \cap \mathrm{adj}(v_i)$		
1	$\{C\}$	$\emptyset \cap \{C\}$	$= \emptyset$	complete
2	$\{A, D, F\}$	$\{A\} \cap \{A, D, F\}$	$= \{A\}$	complete
3	$\{C, B, E, F\}$	$\{A, C\} \cap \{C, B, E, F\}$	$= \{C\}$	complete
4	$\{G, C, D, E, H\}$	$\{A, C, D\} \cap \{G, C, D, E, H\}$	$= \{C, D\}$	complete
5	$\{B, D, F, H\}$	$\{A, C, D, F\} \cap \{B, D, F, H\}$	$= \{D, F\}$	complete
6	$\{D, E\}$	$\{A, C, D, F, E\} \cap \{D, E\}$	$= \{D, E\}$	complete
7	$\{F, E\}$	$\{A, C, D, F, E, B\} \cap \{F, E\}$	$= \{F, E\}$	complete
8	$\{F\}$	$\{A, C, D, F, E, B, H\} \cap \{F\}$	$= \{F\}$	complete

α is a perfect ordering

Running Intersection Property

Let $G = (V, E)$ be an undirected graph with p cliques. An ordering of these cliques has the *running intersection property (RIP)*, if for every $j > 1$ there exists an $i < j$ such that:

$$C_j \cap \left(C_1 \cup \cdots \cup C_{j-1}\right) \subseteq C_i$$

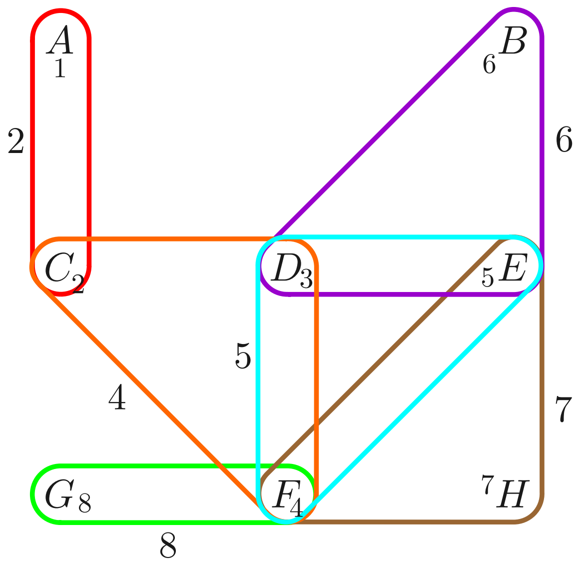


$\xi = \langle C_1,\ C_2,\ C_3,\ C_4,\ C_5,\ C_6 \rangle$

j				i
2	$C_2 \cap C_1$	$= \{C\}$	$\subseteq C_1$	1
3	$C_3 \cap (C_1 \cup C_2)$	$= \{D, F\}$	$\subseteq C_2$	2
4	$C_4 \cap (C_1 \cup C_2 \cup C_3)$	$= \{D, E\}$	$\subseteq C_3$	3
5	$C_5 \cap (C_1 \cup C_2 \cup C_3 \cup C_4)$	$= \{E, F\}$	$\subseteq C_3$	3
6	$C_6 \cap (C_1 \cup C_2 \cup C_3 \cup C_4 \cup C_5)$	$= \{F\}$	$\subseteq C_5$	5

ξ has running intersection property

If a node ordering α of an undirected graph $G = (V, E)$ is perfect and the cliques of G are ordered according to the highest rank (w. r. t. α) of the containing nodes, then this clique ordering has RIP.

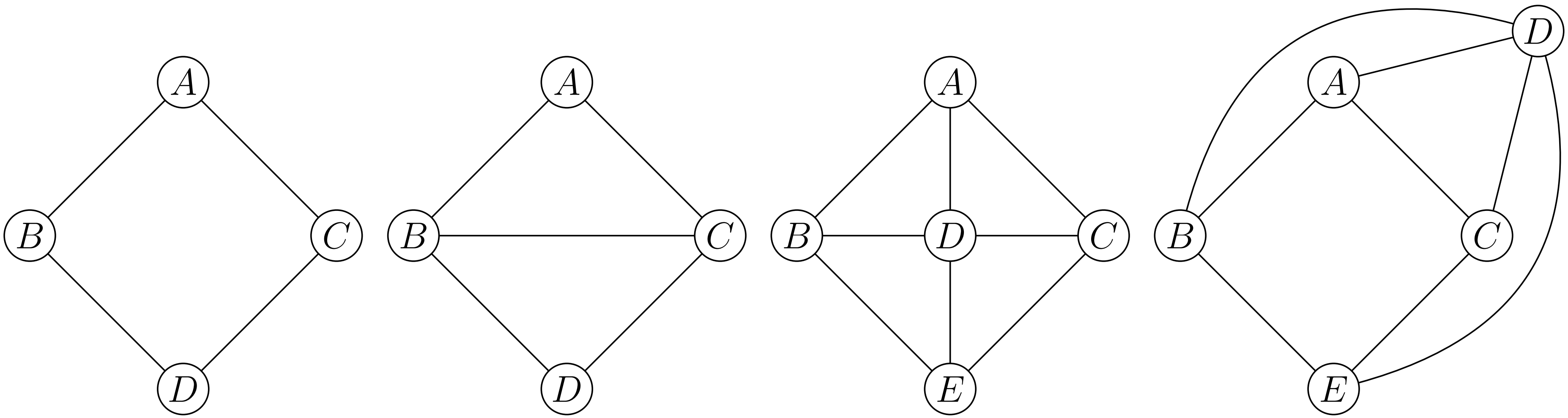


Clique	Rank		
$\{A, C\}$	$\max\{\alpha(A), \alpha(C)\}$	$= 2$	$\rightarrow C_1$
$\{C, D, F\}$	$\max\{\alpha(C), \alpha(D), \alpha(F)\}$	$= 4$	$\rightarrow C_2$
$\{D, E, F\}$	$\max\{\alpha(D), \alpha(E), \alpha(F)\}$	$= 5$	$\rightarrow C_3$
$\{B, D, E\}$	$\max\{\alpha(B), \alpha(D), \alpha(E)\}$	$= 6$	$\rightarrow C_4$
$\{F, E, H\}$	$\max\{\alpha(F), \alpha(E), \alpha(H)\}$	$= 7$	$\rightarrow C_5$
$\{F, G\}$	$\max\{\alpha(F), \alpha(G)\}$	$= 8$	$\rightarrow C_6$

How to get a perfect ordering?

Triangulated Graph

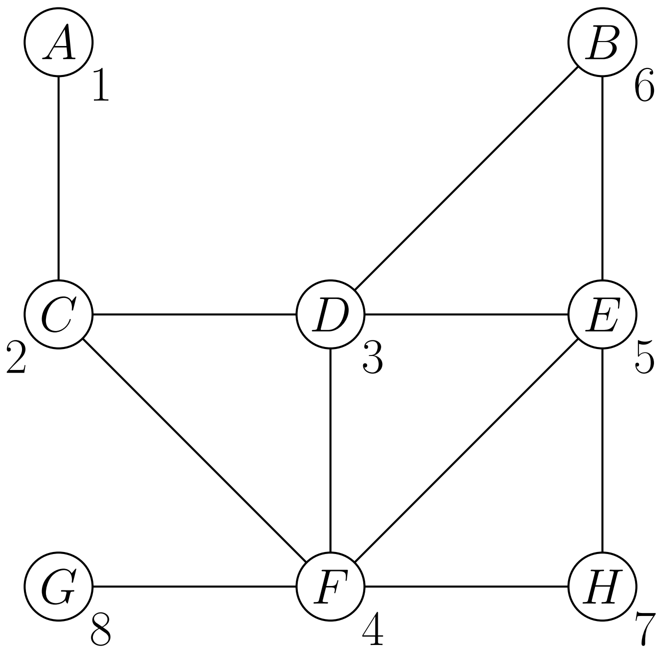
An undirected graph is called *triangulated* if every simple loop (i. e. path with identical start and end node but with any other node occurring at most once) of length greater 3 has a chord.



not triangulated — triangulated — not triangulated — no chord for $\langle A, B, E, C\rangle$

Maximum Cardinality Search

Let $G = (V, E)$ be an undirected graph. An ordering according *maximum cardinality search (MCS)* is obtained by first assigning 1 to an arbitray node. If n numbers are assigned the node that is connected to most of the nodes already numbered gets assigned number $n + 1$.



3 can be assigned to D or F

6 can be assigned to H or B

Triangulated Graphs (3)

If an undirected graph is triangulated, then the ordering obtained by MCS is perfect.

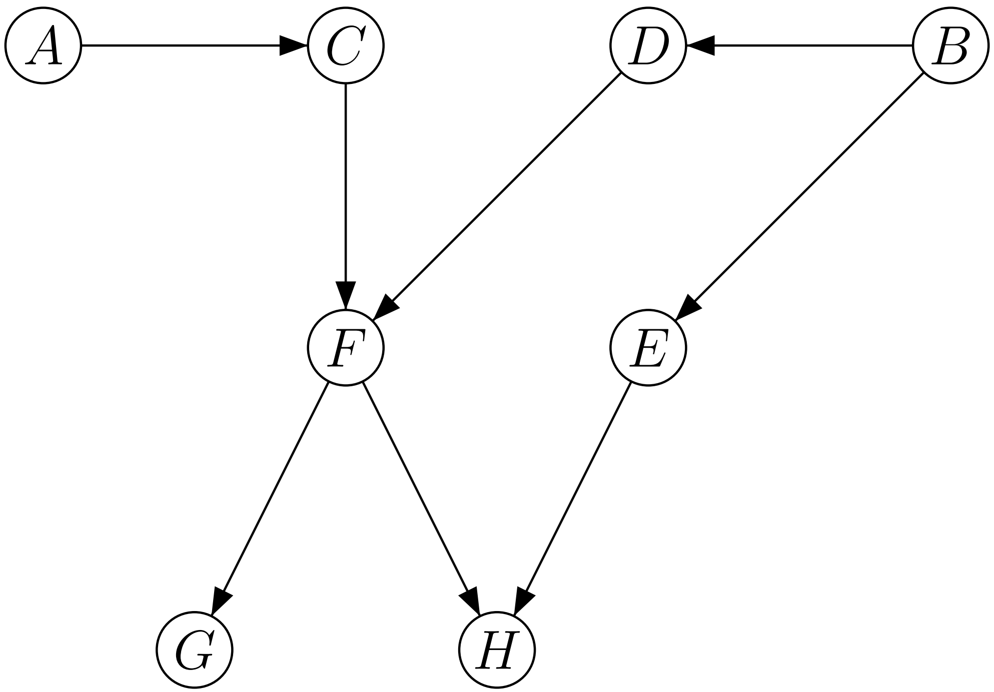
To check whether a graph is triangulated is efficient to implement. The optimization problem that is related to the triangulation task is NP-hard. However, there are good heuristics.

Moral Graph (Repetition)

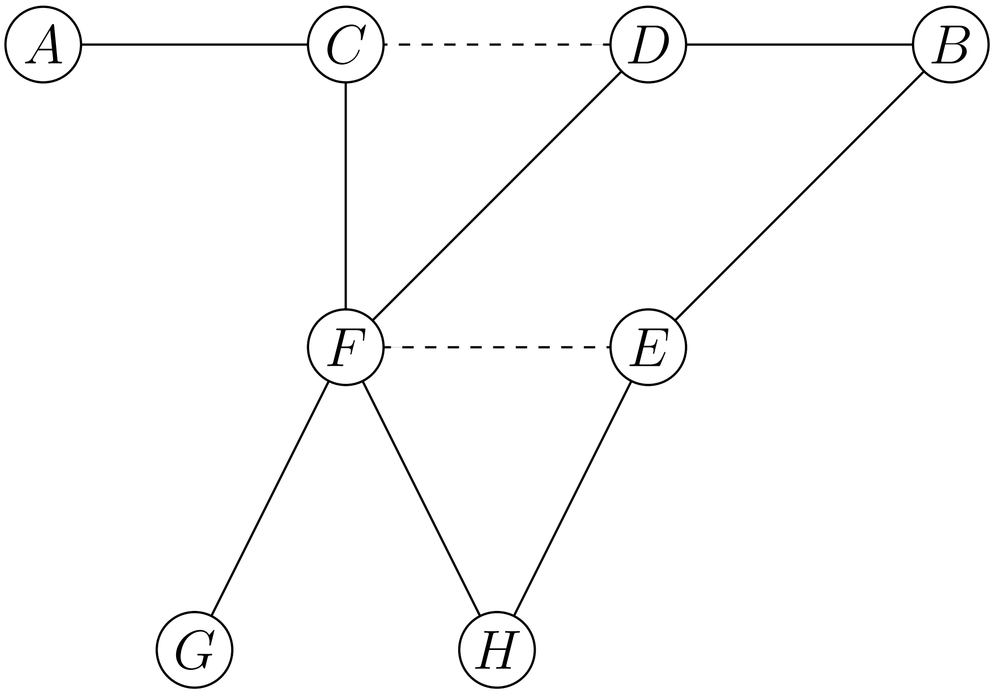
Let $G = (V, E)$ be a directed acyclic graph. If $u, w \in W$ are parents of $v \in V$ connect u and w with an (arbitrarily oriented) edge. After the removal of all edge directions the resulting graph $G_m = (V, E')$ is called the *moral graph* of G.

Join-Tree Construction (1)

Given directed graph.

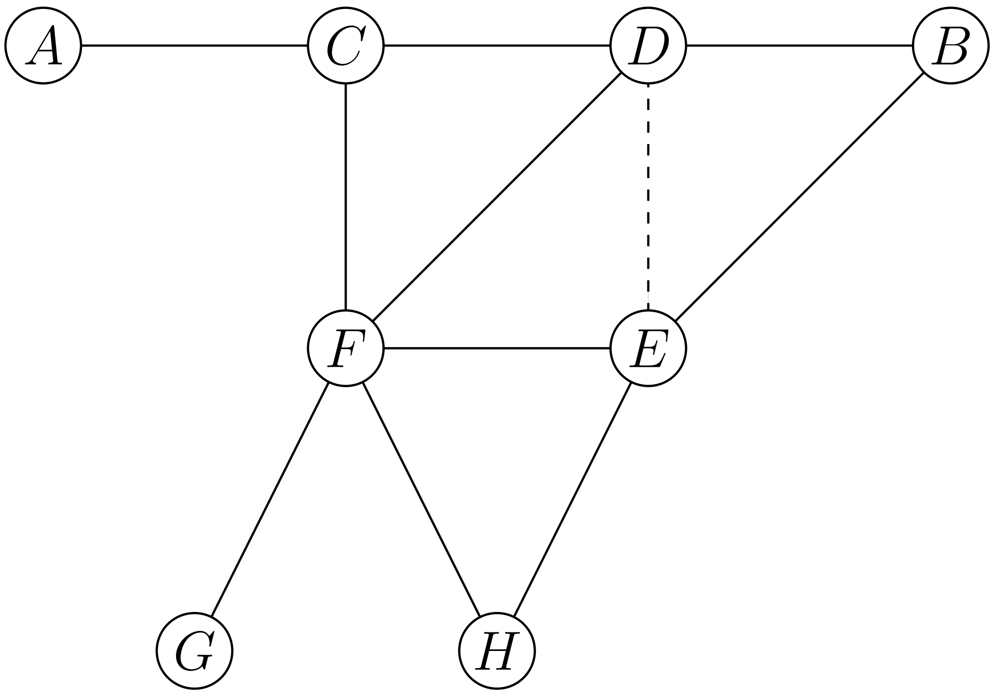


Join-Tree Construction (2)



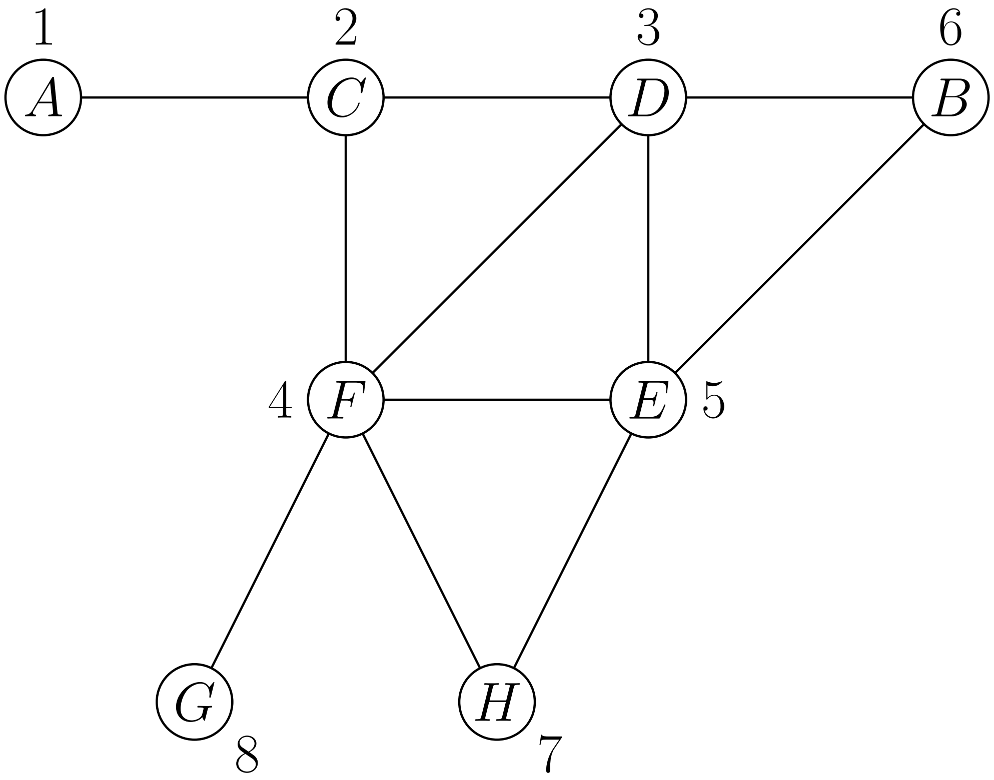
- Moral graph

Join-Tree Construction (3)



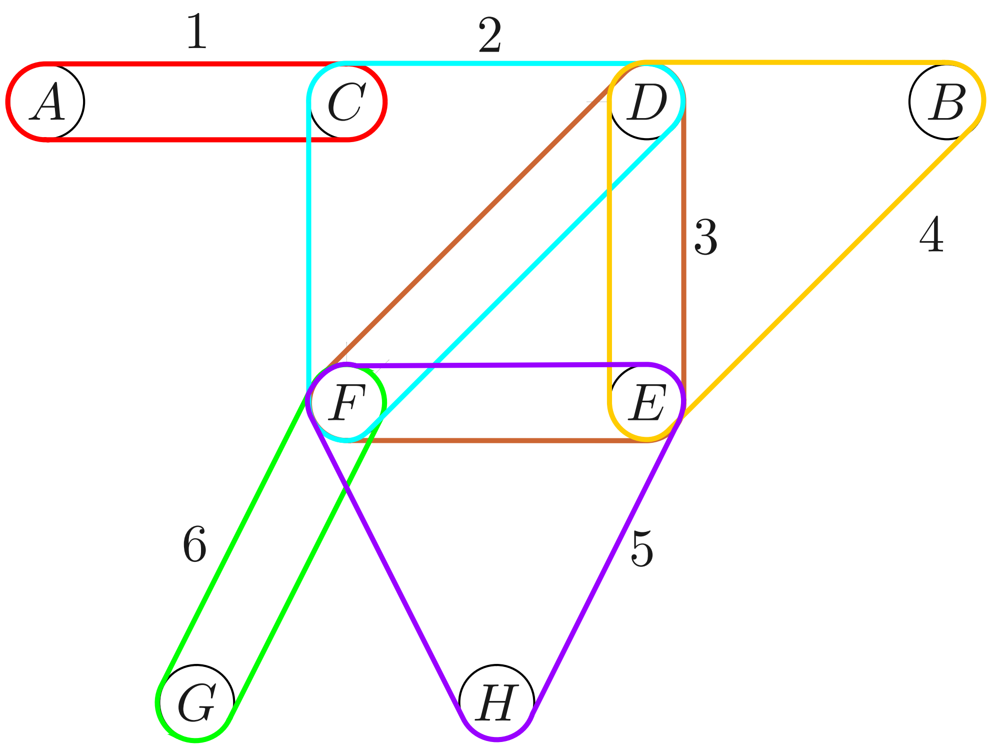
- Moral graph
- Triangulated graph

Join-Tree Construction (4)



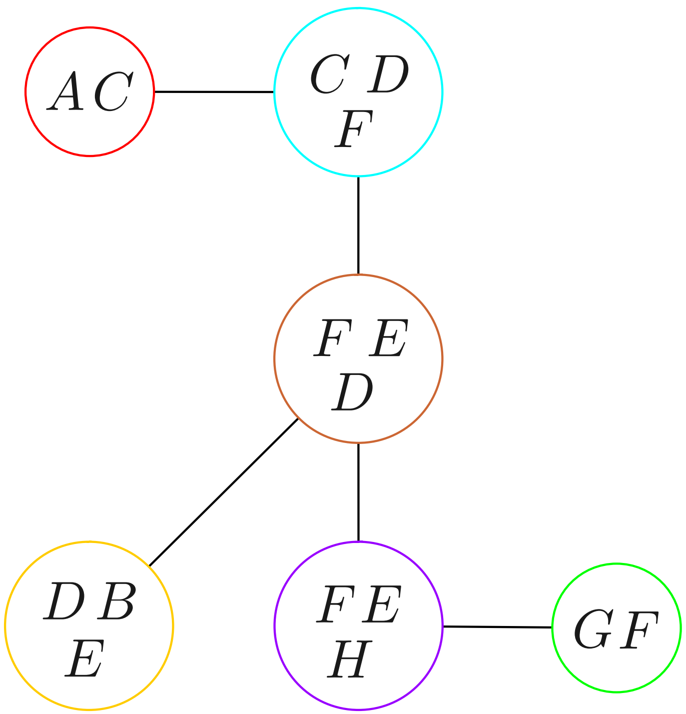
- Moral graph
- Triangulated graph
- MCS yields perfect ordering

Join-Tree Construction (5)



- Moral graph
- Triangulated graph
- MCS yields perfect ordering
- Clique order has RIP

Join-Tree Construction (6)



- Moral graph
- Triangulated graph
- MCS yields perfect ordering
- Clique order has RIP
- Form a join-tree

  Two cliques can be connected if they have a non-empty intersection. The generation of the tree follows the RIP. In case of a tie, connect cliques with the largest intersection. (e. g. DBE—FED instead of DBE—CFD) Break remaining ties arbitrarily.

Example: Expert Knowledge

Qualitative knowledge:

Metastatic cancer is a possible cause of brain tumor, and is also an explanation for increased total serum calcium. In turn, either of these could explain a patient falling into a coma. Severe headache is also possibly associated with a brain tumor.

Special case:

The patient has heavy headache.

Query:

Will the patient fall into coma?

Example: Choice of State Space

	Attribute	Possible Values	
A	metastatic cancer	$\mathrm{dom}(A) = \{a_1, a_2\}$	$\cdot_1 =$ existing
B	increased total serum calcium	$\mathrm{dom}(B) = \{b_1, b_2\}$	$\cdot_2 =$ notexisting
C	brain tumor	$\mathrm{dom}(C) = \{c_1, c_2\}$	
D	coma	$\mathrm{dom}(D) = \{d_1, d_2\}$	
E	severe headache	$\mathrm{dom}(E) = \{e_1, e_2\}$	

Exhaustive state space:

$$\Omega = \mathrm{dom}(A) \times \mathrm{dom}(B) \times \mathrm{dom}(C) \times \mathrm{dom}(D) \times \mathrm{dom}(E)$$

Marginal and conditional probabilities are of interest for the user!

Example: Qualitative Knowledge

$$\left.\begin{aligned} P(e_1 \mid c_1) &= 0.8 \\ P(e_1 \mid c_2) &= 0.6 \end{aligned}\right\} \text{headaches common, but more common if tumor present}$$

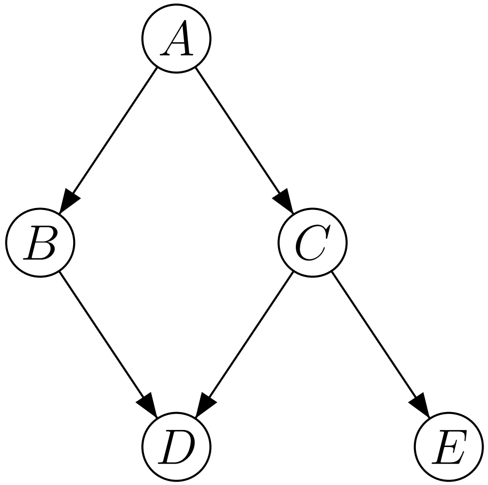
$$\left.\begin{aligned} P(d_1 \mid b_1, c_1) &= 0.8 \\ P(d_1 \mid b_1, c_2) &= 0.8 \\ P(d_1 \mid b_2, c_1) &= 0.8 \\ P(d_1 \mid b_2, c_2) &= 0.05 \end{aligned}\right\} \text{coma rare but common, if either cause is present}$$

$$\left.\begin{aligned} P(b_1 \mid a_1) &= 0.8 \\ P(b_1 \mid a_2) &= 0.2 \end{aligned}\right\} \begin{array}{c}\text{increased calcium uncommon,} \\ \text{but common consequence of metastases}\end{array}$$

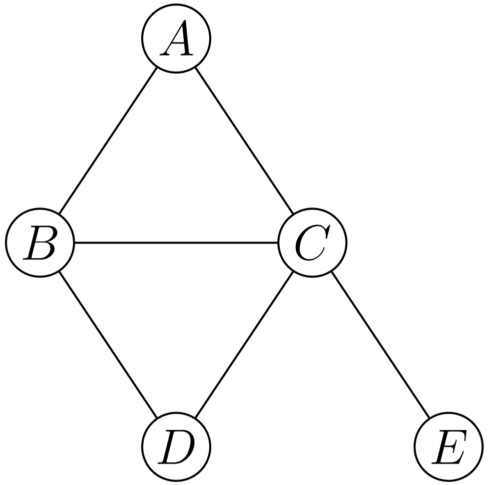
$$\left.\begin{aligned} P(c_1 \mid a_1) &= 0.2 \\ P(c_1 \mid a_2) &= 0.05 \end{aligned}\right\} \text{brain tumor rare, and uncommon consequence of metastases}$$

$$\left.P(a_1) = 0.2 \right\} \text{incidence of metastatic cancer in relevant clinic}$$

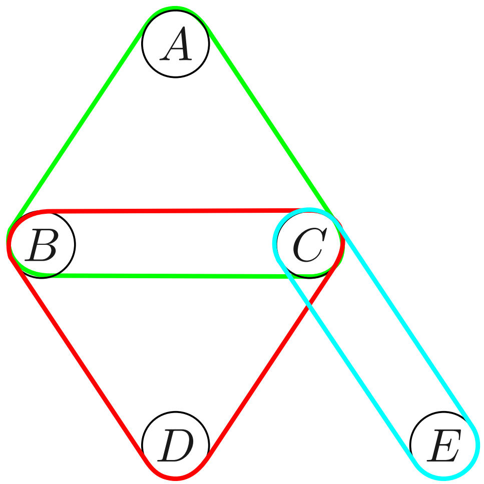
Example: Metastatic Cancer



Dependencies



Moralization/Triangulation



MCS, hyper graph

Example (2)

Quantitative knowledge:

(a, b, c)	$P(a, b, c)$
a_1, b_1, c_1	0.032
a_2, b_1, c_1	0.008
$\vdots$	$\vdots$
a_2, b_2, c_2	0.608

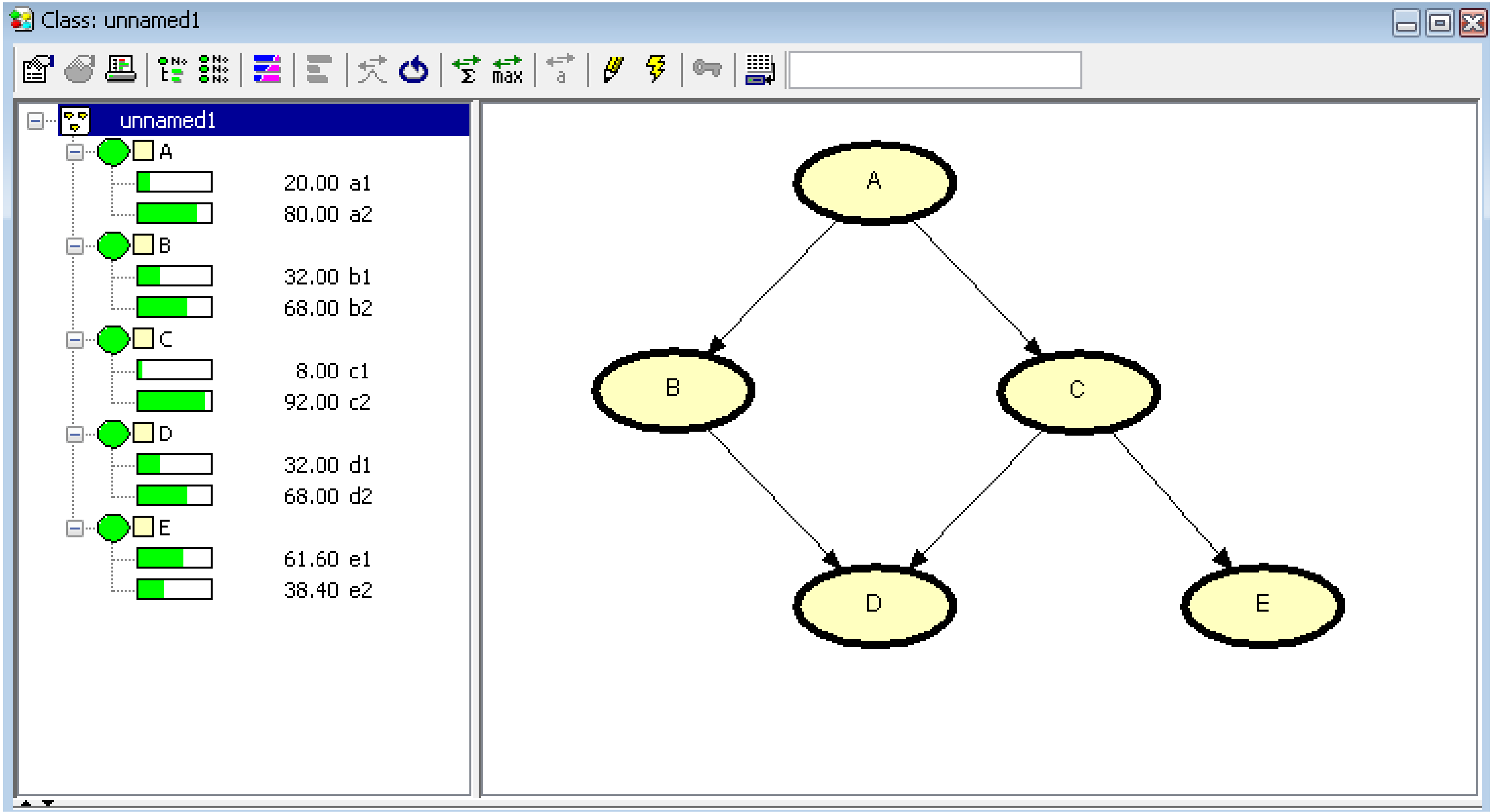
(b, c, d)	$P(b, c, d)$
b_1, c_1, d_1	0.032
b_2, c_1, d_1	0.032
$\vdots$	$\vdots$
b_2, c_2, d_2	0.608

(c, e)	$P(c, e)$
c_1, e_1	0.064
c_2, e_1	0.552
c_1, e_2	0.016
c_2, e_2	0.368

Decomposition:

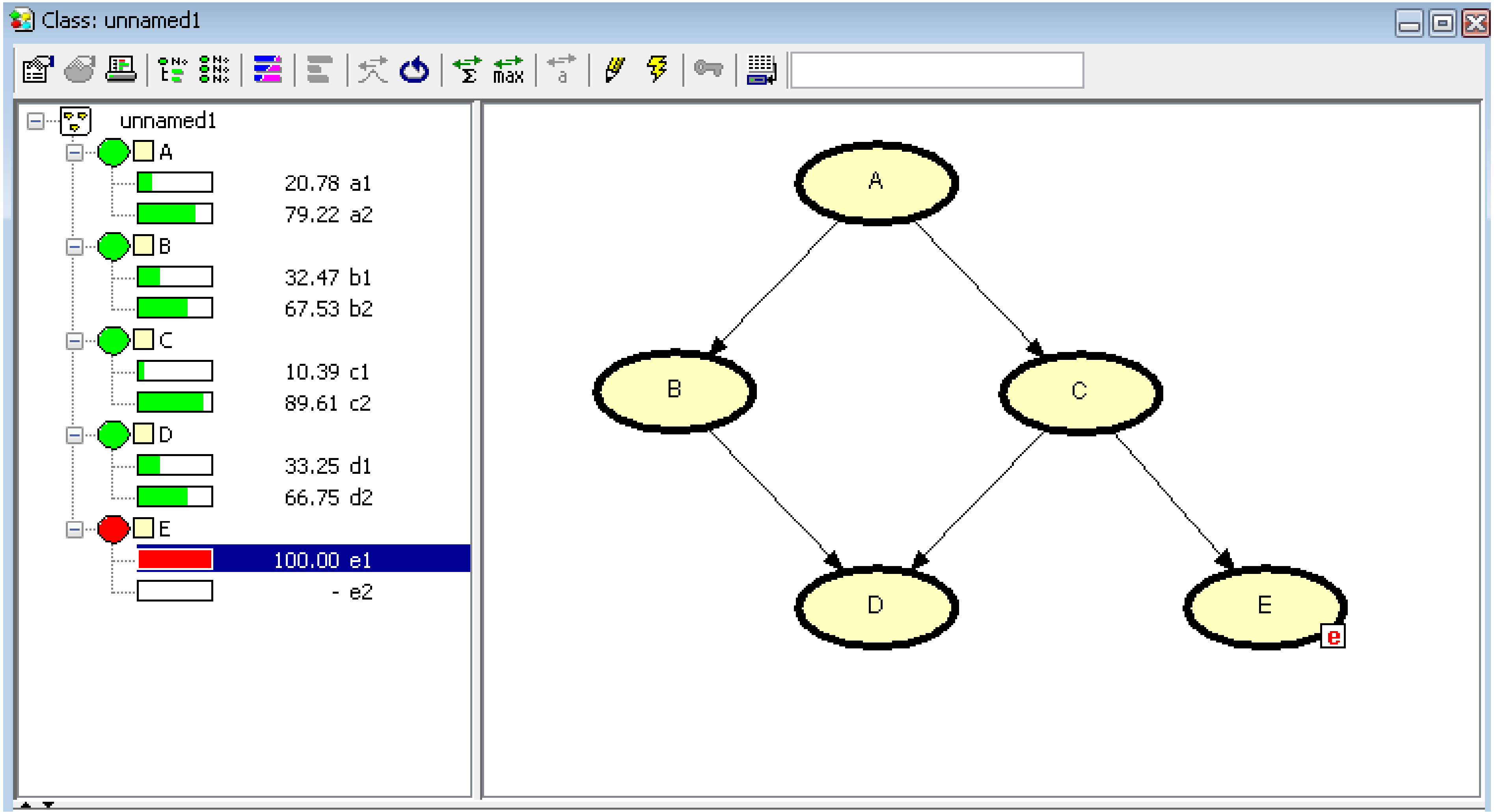
$$\begin{aligned} P(A, B, C, D, E) &= P(A)P(B \mid A)P(C \mid A)P(D \mid BC)P(E \mid C) \\ &= \frac{P(A, B)P(B, C, D), P(C, E)}{P(BC)P(C)} \end{aligned}$$

Example (3)



Marginal distributions in the HUGIN tool.

Example (4)



Conditional marginal distributions with evidence $E = e_1$