OTTO VON GUERICKE UNIVERSITÄT MAGDEBURG

INF FAKULTÄT FÜR INFORMATIK

Fuzzy Systems
Fuzzy Sets and Fuzzy Logic

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Motivation

Every day humans use imprecise linguistic terms
e.g. big, *fast*, *about 12 o'clock*, *old*, etc.

All complex human actions are decisions based on such concepts:

- driving and parking a car,
- financial/business decisions,
- law and justice,
- giving a lecture,
- listening to the professor/tutor.

So, these terms and the way they are processed play a crucial role.

Computers need a mathematical model to express and process such complex semantics.

Concepts of classical mathematics are inadequate for such models.

Lotfi Asker Zadeh

Classes of objects in the real world do not have precisely defined criteria of membership.

Such imprecisely defined "classes" play an important role in human thinking,

Particularly in domains of pattern recognition, communication of information, and abstraction.

Lotfi A. Zadeh's Principle of Incompatibility

> *"Stated informally, the essence of this principle is that as the complexity of a system increases, our ability to make precise and yet significant statements about its behavior diminishes until a threshold is reached beyond which precision and significance (or relevance) become almost mutually exclusive characteristics."*

Fuzzy sets/fuzzy logic are used as mechanism for abstraction of unnecessary or too complex details.

Example – The Sorites Paradox

If a sand dune is small, adding one grain of sand to it leaves it small.
A sand dune with a single grain is small.

Hence all sand dunes are small.

Paradox comes from all-or-nothing treatment of *small*.

Degree of truth of “heap of sand is small” decreases by adding one grain after another.

Certain number of words refer to continuous numerical scales.

Example – The Sorites Paradox

How many grains of sand has a sand dune at least?

Statement $A(n)$: "n grains of sand are a sanddune."

Let $d_n = T(A(n))$ denote "degree of acceptance" for $A(n)$.

Then

$$0 = d_0 \leq d_1 \leq \ldots \leq d_n \leq \ldots \leq 1$$

can be seen as truth values of a **many valued logic**.

Toy Example

Consider the notion *bald*:
A man without hair on his head is bald,
a hairy man is not bald.

Usually, *bald* is only partly applicable.

Where to set *baldness/non baldness* threshold?

Fuzzy set theory does not assume any threshold!

Applications of Fuzzy Systems

Control Engineering

Approximate Reasoning

Data Sciences

Rudolf Kruse received IEEE Fuzzy Pioneer Award for „Learning Methods for Fuzzy Systems“ in 2018

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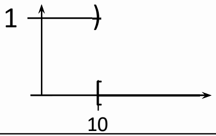
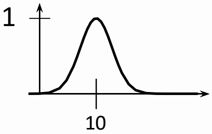
Fuzzy Sets - Basics

Fuzzy sets are generalizations of classical sets

ling. description		model	
all numbers smaller than 10	*objective*		characteristic function of a set
all numbers <u>almost</u> equal to 10	*subjective*		membership function of a “fuzzy set”

Definition

A fuzzy set μ of X is a function from the reference set X to the unit interval, *i.e.* $\mu : X \to [0, 1]$. $F(X)$ represents the set of all fuzzy sets of X, *i.e.* $F(X) := \{\mu \mid \mu : X \to [0, 1]\}$.

Membership Functions

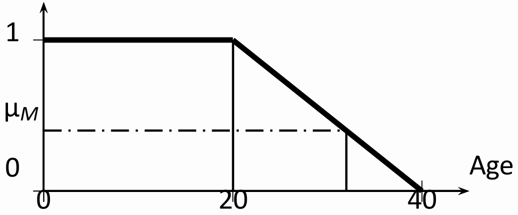
$\mu_M(u) = 1$ reflects full membership in M.

$\mu_M(u) = 0$ expresses absolute non-membership in M.

Sets can be viewed as special case of fuzzy sets where only full membership and absolute non-membership are allowed.

Such sets are called *crisp sets* or Boolean sets.

Membership degrees $0 < \mu_M < 1$ represent *partial membership*.



Representing *young* in "a young person"

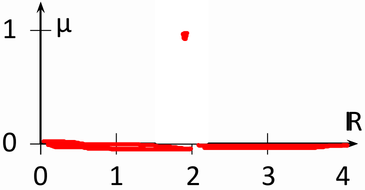
Membership Functions

A Membership function attached to a given linguistic description (such as *young*) depends on the context – it is subjective.

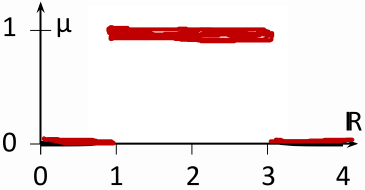
A young retired person is certainly older than a young student.
Even the idea of young student depends on the user.

Membership degrees are fixed only *by convention*:
Unit interval as range of membership grades is arbitrary but easy to use.

Examples for Fuzzy Sets

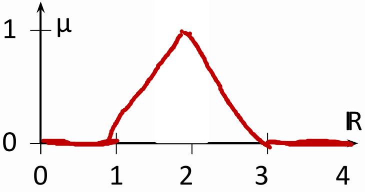


exactly two

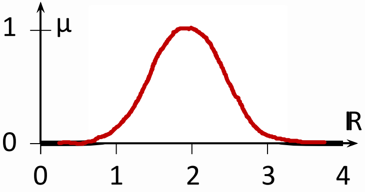


between 1 and 3

Examples for Fuzzy Sets



Approximately 2



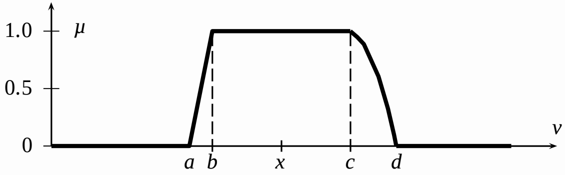
Approximately 2

Exact numerical value has membership degree of 1.

Left: monotonically increasing, right: monotonically decreasing, *i.e.* unimodal function.

Terms like *around* modeled using triangular or Gaussian function.

Example – Velocity of Rotating Hard Disk



Fuzzy set μ characterizing the normal velocity of rotating hard disk.

Let *v* be the velocity of rotating hard disk in revolutions per minute.

Modelling of expert's knowledge:

"It's *impossible* that *v* drops under *a* or exceeds *d* .

"It's highly certain that any value between [*b*, *c*] can occur."

„Otherwise I defined my subjective point of view , I also use my data"

Vertical Representation

So far, fuzzy sets were described by
their characteristic/membership function and
assigning degree of membership $\mu(x)$ to each element $x \in X$.

That is the **vertical representation** of the corresponding fuzzy set, *e.g.* linguistic expression like "about m"

$$\mu_{m,d}(x) = \begin{cases} 1 - \left|\frac{m-x}{d}\right|, & \text{if } m - d \leq x \leq m + d \\ 0, & \text{otherwise,} \end{cases}$$

or "approximately between b and c"

$$\mu_{a,b,c,d}(x) = \begin{cases} \frac{x-a}{b-a}, & \text{if } a \leq x < b \\ 1, & \text{if } b \leq x \leq c \\ \frac{x-d}{c-d}, & \text{if } c < x \leq d \\ 0, & \text{if } x < a \text{ or } x > d. \end{cases}$$

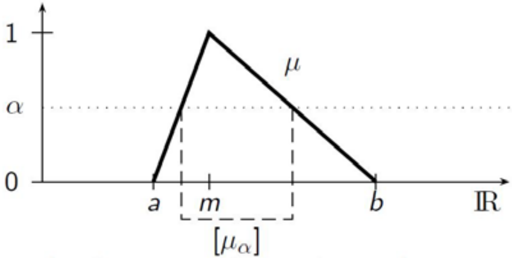
Level Sets (cuts) for a Fuzzy set

Let $\mu \in \mathcal{F}(X)$ and $\alpha \in [0,1]$. Then the sets

$$[\mu]_\alpha = \{x \in X \mid \mu(x) \geq \alpha\}, \quad [\mu]_{\underline{\alpha}} = \{x \in X \mid \mu(x) > \alpha\}$$

are called the α-*cut* and *strict* α-*cut* of μ.

An Example



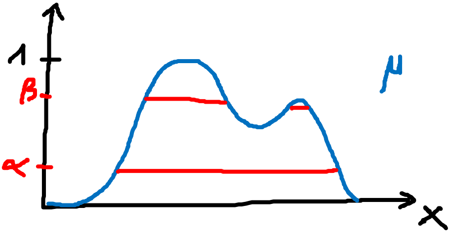
Let μ be triangular function on $\mathbb{R}$ as shown above.

α-cut of μ can be constructed by

1. drawing horizontal line parallel to x-axis through point $(0, \alpha)$,
2. projecting this section onto x-axis.

$$[\mu]_\alpha = \begin{cases} [a + \alpha(m - a), b - \alpha(b - m)], & \text{if } 0 < \alpha \leq 1, \\ \mathbb{R}, & \text{if } \alpha = 0. \end{cases}$$

Properties of α-cuts I



Theorem

Let $\mu \in \mathcal{F}(X)$, $\alpha \in [0,1]$ *and* $\beta \in [0,1]$.

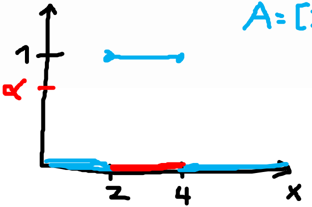
(a) $[\mu]_0 = X$,

(b) $\alpha < \beta \implies [\mu]_\alpha \supseteq [\mu]_\beta$,

(c) $\bigcap_{\alpha:\alpha<\beta} [\mu]_\alpha = [\mu]_\beta$.

$A = [2,4]$

Characteristic function



Let $A \subseteq X, \chi_A : X \to [0,1]$

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{otherwise} \end{cases}$$

Then $[\chi_A]_\alpha = A$ for $0 < \alpha \leq 1$.

χ_A is called indicator function or characteristic function of A.

Properties of α-cuts II

Theorem (Representation Theorem)
Let $\mu \in \mathcal{F}(X)$. Then

$$\mu(x) = \sup_{\alpha \in [0,1]} \left\{ \min(\alpha, \chi_{[\mu]_\alpha}(x)) \right\}$$

where $\chi_{[\mu]_\alpha}(x) = \begin{cases} 1, & \textit{if } x \in [\mu]_\alpha \\ 0, & \textit{otherwise.} \end{cases}$

So, fuzzy set can be obtained as upper envelope of its α-cuts.

Simply draw α-cuts parallel to horizontal axis in height of α.

In applications it is recommended to select finite subset $L \subseteq [0, 1]$ of relevant degrees of membership.

They must be semantically distinguishable.

That is, fix level sets of fuzzy sets to characterize only for these levels.

System of Sets

In this manner we obtain **system of sets**

$$\mathcal{A} = (A_\alpha)_{\alpha \in L}, \quad L \subseteq [0,1], \quad \mathrm{card}(L) \in \mathbb{N}.$$

$\mathcal{A}$ must satisfy consistency conditions for $\alpha, \beta \in L$:

(a) $0 \in L \Longrightarrow A_0 = X,$ (fixing of reference set)

(b) $\alpha < \beta \Longrightarrow A_\alpha \supseteq A_\beta.$ (monotonicity)

This induces fuzzy set

$$\mu_{\mathcal{A}} : X \to [0,1],$$
$$\mu_{\mathcal{A}}(x) = \sup_{\alpha \in L} \left\{ \min(\alpha, \chi_{A_\alpha}(x)) \right\}.$$

If L is not finite but comprises all values $[0,1]$, then μ must satisfy

(c) $\bigcap\limits_{\alpha:\alpha<\beta} A_\alpha = A_\beta.$ (condition for continuity)

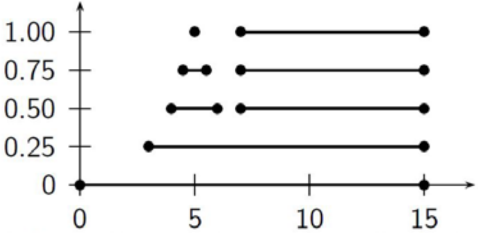
"Approximately 5 or greater than or equal to 7"

An Exemplary Horizontal View

Suppose that $X = [0, 15]$.

An expert chooses $L = \{0, 0.25, 0.5, 0.75, 1\}$ and α-cuts:

- $A_0 \quad = [0, 15]$,
- $A_{0.25} = [3, 15]$,
- $A_{0.5} \ = [4, 6] \cup [7, 15]$,
- $A_{0.75} = [4.5, 5.5] \cup [7, 15]$,
- $A_1 \quad = \{5\} \cup [7, 15]$.



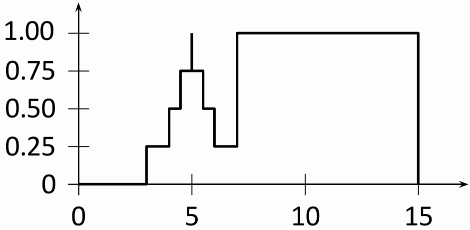
The family $(A_\alpha)_{\alpha \in L}$ of sets induces upper shown fuzzy set.

“Approximately 5 or greater than or equal to 7”

An Exemplary Vertical View

μ_A is obtained as upper envelope of the family A of sets.

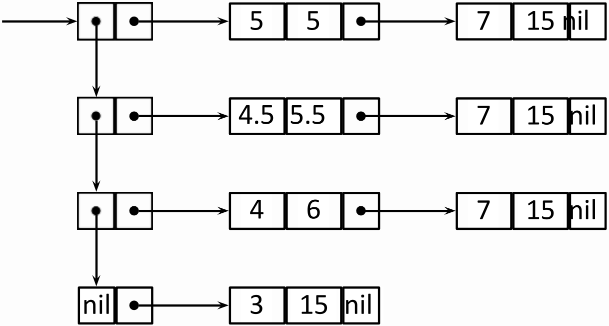
The difference between horizontal and vertical view is obvious:



The horizontal representation is easier to process in computers.

Also, restricting the domain of x-axis to a discrete set is usually done.

Horizontal Representation in the Computer

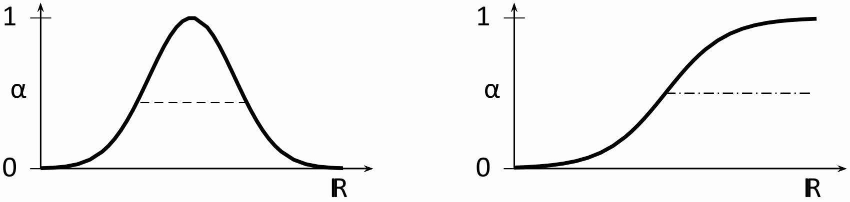


Fuzzy sets are usually stored as chain of linear lists.

A finite union of closed intervals is stored by their bounds.

This data structure is appropriate for arithmetic operators.

Convex Fuzzy Sets



A fuzzy set $\mu \in F(\mathrm{IR})$ *is convex if and only if*

$$\mu(\lambda x_1 + (1 - \lambda)x_2) \geq \min\{\mu(x_1), \mu(x_2)\}$$

for all $x_1, x_2 \in \mathrm{IR}$ *and all* $\lambda \in [0, 1]$.

Fuzzy Logic

The Traditional or Aristotlelian Logic

What is logic about? Different schools speak different languages!

There are raditional, linguistic, psychological, epistemological and mathematical schools.

Traditional logic has been founded by Aristotle (384-322 B.C.).

Aristotlelian logic can be seen as formal approach to human reasoning.

It's still used today in Artificial Intelligence for knowledge representation and reasoning about knowledge.

Detail of "The School of Athens" by R. Sanzio (1509) showing Plato (left) and his student Aristotle (right).

Classical Logics is intuitive

Logics study methods/principles of **reasoning**.

The most famous logic is the **propositional calculus.**

A **proposition** can be (only) *true* or *false,* the calculus uses **connectives** such as „and“ (∧), „or“(∨), „not“(¬), „imply“(→).

The calculus uses **inference rules** (like modus ponens):

Premise 1: If it's raining then it's cloudy.
Premise 2: It's raining.
Conclusion: It's cloudy.

But formalization of Propositional Logic is tricky

Formal Language (Symbols, Operators, Well-formed formulas, formation rules,..)

Truth Functions and Truth Tables

Tautologies (true for all possible truth-value assignments)

Deduction System (modus ponens, resolution, modus tollens,…)

Desirable **Meta Theoretic Properties** (Completeness, Soundness, Consistency, Truth Functionality)

Many-valued logics consider more than two truth-values, in the simplest form the values true, false, and indeterminate

Boolean Algebra

The propositional logic based on finite set of logic variables is isomorphic to **finite set theory**.

Both of these systems are isomorphic to a finite **Boolean algebra**.

A *Boolean algebra* on a set B is defined as quadruple $\mathcal{B} = (B, +, \cdot, \overline{})$ where B has at least two elements (bounds) 0 and 1, $+$ and $\cdot$ are binary operators on B, and $\overline{}$ is a unary operator on B for which the following properties hold.

Properties of Boolean Algebras I

(B1) Idempotence	$a + a = a$	$a \cdot a = a$
(B2) Commutativity	$a + b = b + a$	$a \cdot b = b \cdot a$
(B3) Associativity	$(a + b) + c = a + (b + c)$	$(a \cdot b) \cdot c = a \cdot (b \cdot c)$
(B4) Absorption	$a + (a \cdot b) = a$	$a \cdot (a + b) = a$
(B5) Distributivity	$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$	$a + (b \cdot c) = (a + b) \cdot (a + c)$
(B6) Universal Bounds	$a + 0 = a,\ a + 1 = 1$	$a \cdot 1 = a,\ a \cdot 0 = 0$
(B7) Complementary	$a + \overline{a} = 1$	$a \cdot \overline{a} = 0$
(B8) Involution	$\overline{\overline{a}} = a$	
(B9) Dualization	$\overline{a + b} = \overline{a} \cdot \overline{b}$	$\overline{a \cdot b} = \overline{a} + \overline{b}$

Properties (B1)-(B4) are common to every **lattice**,

i.e. a Boolean algebra is a distributive (B5), bounded (B6), and complemented (B7)-(B9) lattice,

i.e. every Boolean algebra can be characterized by a partial ordering on a set, *i.e.* $a \leq b$ if $a \cdot b = a$ or, alternatively, if $a + b = b$.

Set Theory, Boolean Algebra, PropositionalLogic

Every theorem in one theory has a counterpart in each other theory.

Counterparts can be obtained applying the following substitutions:

Meaning	Set Theory	Boolean Algebra	Prop. Logic
values	2^X	B	$\mathrm{L}(V)$
“meet”/“and”	$\cap$	$\cdot$	$\wedge$
“join”/“or”	$\cup$	$+$	$\vee$
“complement”/“not”	^{c}		$\neg$
identity element	X	1	1
zero element	$\emptyset$	0	0
partial order	$\subseteq$	$\leq$	$\rightarrow$

power set 2^X, set of logic variables V, set of all combinations $\mathrm{L}(V)$ of truth values of V

The Basic Principle of Classical Logic

The Principle of Bivalence:

"Every proposition is either true or false."

It has been formally developed by Tarski.

Alfred Tarski (1902-1983)

Łukasiewicz suggested to replace it by
The Principle of Valence:

"Every proposition has a truth value."

Propositions can have intermediate truth value,
expressed by a number from the unit interval [0, 1].

Jan Łukasiewicz (1878-1956)

Three-valued Logics

A 2-valued logic can be extended to a 3-valued logic *in several ways*, *i.e.* different three-valued logics have been well established:

truth, falsity, indeterminacy are denoted by 1, 0, and 1/2, resp.

The negation $\neg a$ is defined as $1 - a$, *i.e.* $\neg 1 = 0$, $\neg 0 = 1$ and $\neg 1/2 = 1/2$.

Other primitives, *e.g.* $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, differ from logic to logic.

Five well-known three-valued logics (named after their originators) are defined in the following.

Primitives of Some Three-valued Logics

		Łukasiewicz				Bochvar				Kleene				Heyting				Reichenbach			
a	b	$\wedge$	$\vee$	$\rightarrow$	$\leftrightarrow$	$\wedge$	$\vee$	$\rightarrow$	$\leftrightarrow$	$\wedge$	$\vee$	$\rightarrow$	$\leftrightarrow$	$\wedge$	$\vee$	$\rightarrow$	$\leftrightarrow$	$\wedge$	$\vee$	$\rightarrow$	$\leftrightarrow$
0	0	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
0	$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	1	0	0	$\frac{1}{2}$	1	$\frac{1}{2}$
0	1	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0
$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$	1	1
$\frac{1}{2}$	1	$\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$
1	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0
1	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

All of them fully conform the usual definitions for $a, b \in \{0, 1\}$.
They differ from each other only in their treatment of $1/2$.
Question: Do they satisfy the law of contradiction ($a \wedge \neg a = 0$) and the law of excluded middle ($a \vee \neg a = 1$)?

n-valued Logics

After the three-valued logics: generalizations to n-valued logics for arbitrary number of truth values $n \geq 2$.

In the 1930s, various n-valued logics were developed.

Usually truth values are assigned by rational number in $[0, 1]$.

Key idea: uniformly divide $[0, 1]$ into n truth values.

Definition
The set T_n of truth values of an n-valued logic is defined as

$$T_n = \left\{0 = \frac{0}{n-1}, \frac{1}{n-1}, \frac{2}{n-1}, \ldots, \frac{n-2}{n-1}, \frac{n-1}{n-1} = 1\right\}.$$

These values can be interpreted as degree of truth.

Primitives in *n*-valued Logics

Łukasiewicz proposed first series of n-valued logics for $n \geq 2$. In the early 1930s, he simply generalized his three-valued logic. It uses truth values in T_n and defines primitives as follows:

$$\neg a = 1 - a$$

$$a \wedge b = \min(a, b) \quad a \qquad\qquad a \rightarrow b = \min(1, 1 + b - a)$$

$$a \vee b = \max(a, b) \qquad\qquad a \leftrightarrow b = 1 - |a - b|$$

The n-valued logic of Łukasiewicz is denoted by L_n.

The sequence $(L_2, L_3, \ldots, L_\infty)$ contains the classical two-valued logic L_2 and an infinite-valued logic L_∞ (rational **countable** values T_∞).

The infinite-valued logic L_1 (**standard Łukasiewicz logic**) is the logic with all real numbers in [0, 1] (1 = cardinality of continuum).

Zadeh‘s „Fuzzy Logic" is very simple

In 1965, Zadeh proposed a multivalued logic, called Fuzzy Logic, with values in [0, 1]:

$$\neg a = 1 - a,$$
$$a \wedge b = \min(a, b),$$
$$a \vee b = \max(a, b).$$

The notion of a „Fuzzy Logic" is often use in a much broader sense

Set Operators...

...are defined by using traditional logics operator

Let X be universe of discourse (universalset):

$$A \cap B = \{x \in X \mid x \in A \land x \in B\}$$

$$A \cup B = \{x \in X \mid x \in A \lor x \in B\}$$

$$A^c = \{x \in X \mid x \notin A\} = \{x \in X \mid \neg(x \in A)\}$$

$A \subseteq B$ if and only if $(x \in A) \rightarrow (x \in B)$ for all $x \in X$

Operations on fuzzy set operations use multivalue logic connectives

Standard Fuzzy Set Operators

$(\mu \wedge \mu')(x) := \min\{\mu(x), \mu'(x)\}$ intersection ("AND"),

$(\mu \vee \mu')(x) := \max\{\mu(x), \mu'(x)\}$ union ("OR"),

$\neg\mu(x) := 1 - \mu(x)$ complement ("NOT").

μ is subset of μ' if and only if $\mu \leq \mu'$.

Theorem
$(F(X), \wedge, \vee, \neg)$ *is a complete distributive lattice but no boolean algebra.*

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Fuzzy Set Operators

In set theory, **operators** are defined by **propositional logics operator**

Let X be universal set (often called universe of discourse). Then we define

$$A \cap B = \{x \in X \mid x \in A \wedge x \in B\}$$

$$A \cup B = \{x \in X \mid x \in A \vee x \in B\}$$

$$A^c = \{x \in X \mid x \notin A\} = \{x \in X \mid \neg (x \in A)\}$$

$A \subseteq B$ if and only if $(x \in A) \rightarrow (x \in B)$ for all $x \in X$

Fuzzy Set Operators can be defined by using **multivalues logics operators**

Standard Fuzzy Set Operators

$$(\mu \wedge \mu')(x) := \min\{\mu(x), \mu'(x)\}$$ intersection ("AND"),

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μ is subset of μ' if and only if $\mu \leq \mu'$.

Theorem

$(\mathcal{F}(X), \wedge, \vee, \neg)$ *is a complete distributive lattice, but no Boolean algebra.*

Standard Fuzzy Set Operators

$$(\mu \wedge \mu')(x) := \min\{\mu(x), \mu'(x)\}$$ intersection (“AND”),

$$(\mu \vee \mu')(x) := \max\{\mu(x), \mu'(x)\}$$ union (“OR”),

$$\neg\mu(x) := 1 - \mu(x)$$ complement (“NOT”).

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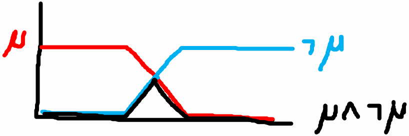


Fuzzy Set Complement

Fuzzy Complement/Fuzzy Negation

Definition

Let X be a given set and $\mu \in \mathcal{F}(X)$. Then the *complement* $\bar{\mu}$ can be defined pointwise by $\bar{\mu}(x) := \sim(\mu(x))$ where $\sim\ : [0,1] \to [0,1]$ satisfies the conditions

$$\sim(0) = 1, \quad \sim(1) = 0$$

and

$$\text{for } x, y \in [0,1],\ x \leq y \implies \sim x \geq \sim y \quad (\sim \text{ is non-increasing}).$$

Abbreviation: $\sim x := \sim(x)$

Strict and Strong Negations

Additional properties may be required

- $x, y \in [0, 1],\ x < y \Longrightarrow \sim x > \sim y$ ($\sim$ is strictly decreasing)
- $\sim$ is continuous
- $\sim\sim x = x$ for all $x \in [0, 1]$ ($\sim$ is involutive)

According to conditions, two subclasses of negations are defined:

Definition
A negation is called *strict* if it is also strictly decreasing and continuous. A strict negation is said to be *strong* if it is involutive,too.

$\sim x = 1 - x^2$, for instance, is strict, not strong, thus not involutive

Families of Negations

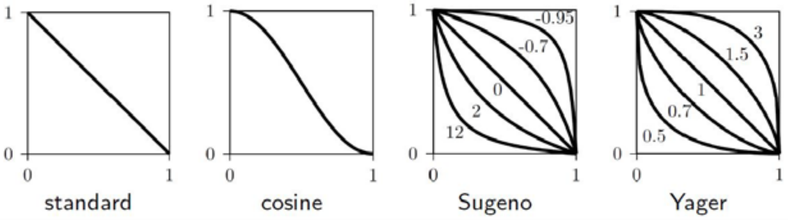
standard negation: $\sim x = 1 - x$

threshold negation: $\sim_\theta(x) = \begin{cases} 1 & \text{if } x \le \theta \\ 0 & \text{otherwise} \end{cases}$

Cosine negation: $\sim x = \frac{1}{2}(1 + \cos(\pi x))$

Sugeno negation: $\sim_\lambda(x) = \frac{1 - x}{1 + \lambda x}, \quad \lambda > -1$

Yager negation: $\sim_\lambda(x) = (1 - x^\lambda)^{\frac{1}{\lambda}}$



Fuzzy Set Intersection and Union

Classical Intersection and Union

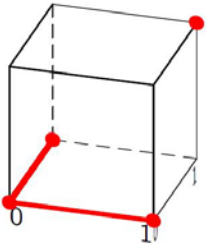
Classical set intersection represents logical conjunction.

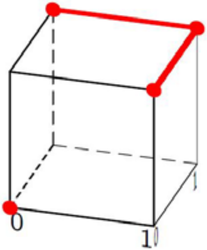
Classical set union represents logical disjunction.

Generalization from $\{0, 1\}$ to $[0, 1]$ as follows:

$x \wedge y$	0	1
0	0	0
1	0	1

$x \vee y$	0	1
0	0	1
1	1	1





Fuzzy Set Intersection and Union

Let A, B be fuzzy subsets of X, *i.e.* $A, B \in \mathcal{F}(X)$.

Their **intersection** and **union** are often defined pointwise using:

$$(A \cap B)(x) = \top(A(x), B(x)) \qquad \text{where} \qquad \top : [0,1]^2 \to [0,1]$$

$$(A \cup B)(x) = \bot(A(x), B(x)) \qquad \text{where} \qquad \bot : [0,1]^2 \to [0,1].$$

Triangular Norms and Conorms

$\top$ is a *triangular norm (t-norm)* $\Longleftrightarrow$ $\top$ satisfies conditions T1-T4

$\bot$ is a *triangular conorm (t-conorm)* $\Longleftrightarrow$ $\bot$ satisfies C1-C4

Identity Law

T1: $\top(x, 1) = x$

C1: $\bot(x, 0) = x$

Commutativity

T2: $\top(x, y) = \top(y, x)$

C2: $\bot(x, y) = \bot(y, x)$

Associativity

T3: $\top(x, \top(y, z)) = \top(\top(x, y), z)$

C3: $\bot(x, \bot(y, z)) = \bot(\bot(x, y), z)$

Monotonicity

T4: $y \leq z$ implies $\top(x, y) \leq \top(x, z)$

C4: $y \leq z$ implies $\bot(x, y) \leq \bot(x, z)$.

Triangular Norms and Conorms II

Both identity law and monotonicity respectively imply

$\forall x \in [0,1] : \top(0,x) = 0,$

$\forall x \in [0,1] : \bot(1,x) = 1,$

For any t-norm $\top : \top(x,y) \leq \min(x,y)$, for any t-conorm $\bot : \bot(x,y) \geq \max(x,y)$.

$x = 1 \Rightarrow T(0,1) = 0$ and

$x \leq 1 \Rightarrow T(x,0) \leq T(1,0) = T(0,1) = 0$

De Morgan Triplet I

For every $\top$ and strong negation $\sim$, one can define *t*-conorm $\perp$ by

$$\perp(x,y) = \sim\top(\sim x, \sim y), \qquad x,y \in [0,1].$$

Additionally, in this case $\top(x,y) = \sim\perp(\sim x, \sim y)$, $x,y \in [0,1]$.

De Morgan Triplet II

Definition

The triplet $(\top, \bot, \sim)$ is called *De Morgan triplet* if and only if

$\top$ is t-norm, $\bot$ is t-conorm, $\sim$ is strong negation,

$\top, \bot$ and $\sim$ satisfy $\bot(x, y) = \sim \top(\sim x, \sim y)$.

In the following, some important De Morgan triplets will be shown,

only the most frequently used and important ones.

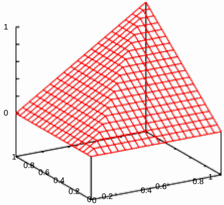
In all cases, the standard negation $\sim x = 1 - x$ is considered.

The Minimum and Maximum I

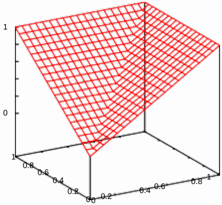
$\top_{\min}(x,y) = \min(x,y), \qquad \bot_{\max}(x,y) = \max(x,y)$

Minimum is the greatest *t*-norm and max is the weakest *t*-conorm.

$\top(x,y) \leq \min(x,y)$ and $\bot(x,y) \geq \max(x,y)$ for any $\top$ and $\bot$



$\top_{\min}$



$\bot_{\max}$

The Special Role of Minimum and Maximum I

$\top_{\min}$ and $\bot_{\max}$ play key role for intersection and union, resp. In a practical sense, they are very simple.

Apart from the identity law, commutativity, associativity and monotonicity, they also satisfy the following properties for all $x, y, z \in [0,1]$:

Distributivity

$\bot_{\max}(x, \top_{\min}(y,z)) = \top_{\min}(\bot_{\max}(x,y), \bot_{\max}(x,z)),$

$\top_{\min}(x, \bot_{\max}(y,z)) = \bot_{\max}(\top_{\min}(x,y), \top_{\min}(x,z))$

Continuity

$\top_{\min}$ and $\bot_{\max}$ are continuous.

The Special Role of Minimum and Maximum II

Strict monotonicity on the diagonal

$x < y$ implies $\top_{\min}(x, x) < \top_{\min}(y, y)$ and $\bot_{\max}(x, x) < \bot_{\max}(y, y)$.

Idempotency

$$\top_{\min}(x, x) = x, \quad \bot_{\max}(x, x) = x$$

Absorption

$$\top_{\min}(x, \bot_{\max}(x, y)) = x, \quad \bot_{\max}(x, \top_{\min}(x, y)) = x$$

Non-compensation

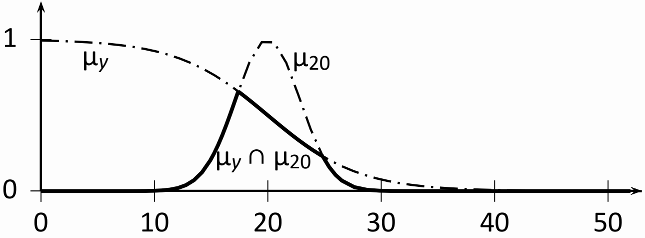
$x < y < z$ imply $\top_{\min}(x, z) \neq \top_{\min}(y, y)$ and $\bot_{\max}(x, z) \neq \bot_{\max}(y, y)$.

The Minimum and Maximum II

$\top_{min}$ and $\perp_{max}$ can be easily processed numerically and visually,

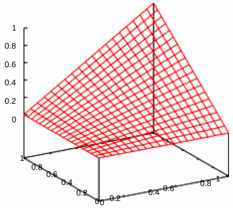
e.g. linguistic values *young* and *approx. 20* described by μ_y , μ_{20}.

$\top_{min}(\mu_y , \mu_{20})$ is shownbelow.

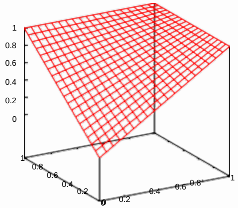


The Product and Probabilistic Sum

$$\top_{\text{prod}}(x, y) = x \cdot y, \qquad \bot_{\text{sum}}(x, y) = x + y - x \cdot y$$



$\top_{\text{prod}}$



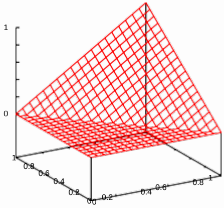
$\bot_{\text{sum}}$

OTTO VON GUERICKE UNIVERSITÄT MAGDEBURG INF FAKULTÄT FÜR INFORMATIK

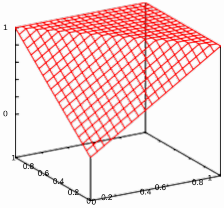
The Łukasiewicz *t*-norm and *t*-conorm

$\top_{Łuka}(x, y) = \max\{0,\ x + y - 1\}$, $\quad\quad \bot_{Łuka}(x, y) = \min\{1,\ x + y\}$

$\top_{Łuka}$, $\bot_{Łuka}$ are also called *bold intersection* and *boundedsum*.



$\top_{Łuka}$



$\bot_{Łuka}$

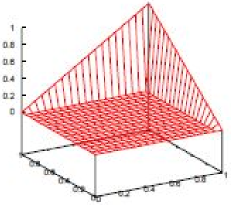
The Drastic Product and Sum

$$\top_{-1}(x,y) = \begin{cases} \min(x,y) & \text{if } \max(x,y) = 1 \\ 0 & \text{otherwise} \end{cases}$$

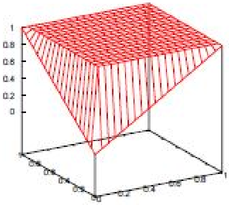
$$\bot_{-1}(x,y) = \begin{cases} \max(x,y) & \text{if } \min(x,y) = 0 \\ 1 & \text{otherwise} \end{cases}$$

$\top_{-1}$ is the weakest t-norm, $\bot_{-1}$ is the strongest t-conorm.

$\top_{-1} \leq \top \leq \top_{\min}, \quad \bot_{\max} \leq \bot \leq \bot_{-1}$ for any $\top$ and $\bot$

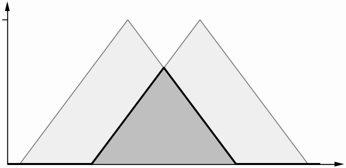


$\top_{-1}$

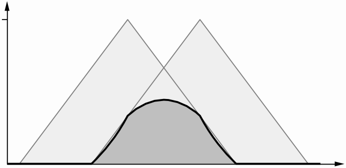


$\bot_{-1}$

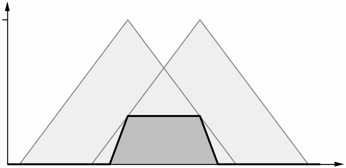
Examples of Fuzzy Intersections



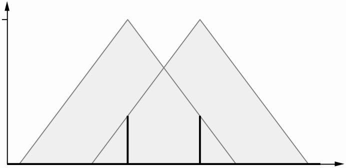
t-norm $T_{\min}$



t-norm T_{prod}



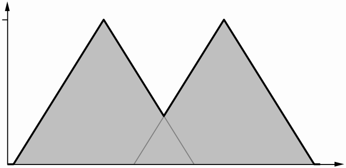
t-norm $T_{\text{Łuka}}$



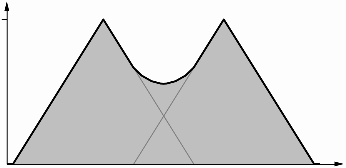
t-norm T_{-1}

Note that all fuzzy intersections are contained within upper left graph and lower right one.

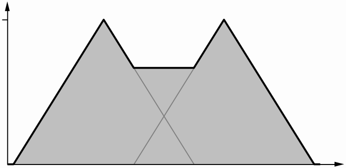
Examples of Fuzzy Unions



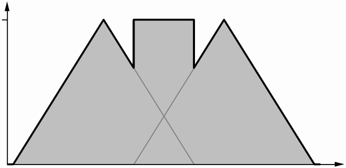
t-conorm $\bot_{\max}$



t-conorm $\bot_{\text{sum}}$



t-conorm $\bot_{\text{Łuka}}$



t-conorm $\bot_{-1}$

Note that all fuzzy unions are contained within upper left graph and lower right one.

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Łukasiewicz Logics

Łukasiewicz proposed a series of *n*-valued logics L_n with truth degrees in T_n

The so called **standard Łukasiewicz logic** has truth degrees in [0, 1]

and uses the following connectives:

$\neg a = 1 - a$	complement
$a \wedge b = \min(a, b)$	weak conjunction
$a \,.\, b = \max(0, a + b - 1)$	strong conjunction
$a \vee b = \max(a, b)$	weak disjunction
$a \times b = \min(1, a+b)$	strong disjunction
$a \rightarrow b = \min(1, 1 + b - a)$	implication
$a \leftrightarrow b = 1 - \lvert a - b \rvert$	biimplication

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Fuzzy Set Operators II

Continuous Archimedian *t*-norms and *t*-conorms

Often it is possible to representation functions with several inputs by a function with only one input , *e.g.*

$$K(x,y) = f^{(-1)}(f(x) + f(y))$$

For a subclass of *t*-norms this is possible. The trick makes calculations simpler.

A *t*-norm $\top$ is called
(a) *continuous* if $\top$ is continuous
(b) *Archimedian* if $\top$ is continuous and $\top(x,x) < x$ for all $x \in]0,1[$.

A *t*-conorm $\perp$ is called
(a) *continuous* if $\perp$ is continuous,

(b) *Archimedian* if $\perp$ is continuous and $\perp(x,x) > x$ for all $x \in]0,1[$.

The concept of a pseudoinverse

Definition

Let $f : [a, b] \to [c, d]$ be a monotone function between two closed subintervals of extended real line. The pseudoinverse function to f is the function $f^{(-1)} : [c, d] \to [a, b]$ defined as

$$f^{(-1)}(y) = \begin{cases} \sup\{x \in [a, b] \mid f(x) < y\} & \text{for } f \text{ non-decreasing,} \\ \sup\{x \in [a, b] \mid f(x) > y\} & \text{for } f \text{ non-increasing.} \end{cases}$$

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Archimedian t-norms

Theorem
A t-norm T *is Archimedian if and only if there exists a strictly decreasing and continuous function* $f : [0,1] \to [0,\infty)$ *with* $f(1) = 0$ *such that*

$$\top(x,y) = f^{(-1)}(f(x) + f(y)) \qquad (1)$$

where

$$f^{(-1)}(x) = \begin{cases} f^{-1}(x) & \text{if } x \leq f(0) \\ 0 & \text{otherwise} \end{cases}$$

is the pseudoinverse of f. *Moreover, this representation is unique up to a positive multiplicative constant.*

$\top$ *is generated by* f if $\top$ has representation (1).

f is called *additive generator* of $\top$.

Additive Generators of t-norms – Examples

Find an additive generator f of $\top_{\text{Łuka}}(x, y) = \max\{x + y - 1,\ 0\}$.

for instance $f_{\text{Łuka}}(x) = 1 - x$

then, $f^{(-1)}_{\text{Łuka}}(x) = \max\{1 - x,\ 0\}$

thus $\top_{\text{Łuka}}(x, y) = f^{(-1)}_{\text{Łuka}}(f_{\text{Łuka}}(x) + f_{\text{Łuka}}(y))$

Find an additive generator f of $\top_{\text{prod}}(x, y) = x \cdot y$.

to be discussed in the exercise

hint: use of logarithmic and exponential function

Archimedian t-conorms

Theorem

A t-conorm $\perp$ Archimedian if and only if there exists a strictly increasing and continuous function $g : [0,1] \to [0,\infty]$ with $g(0) = 0$ such that

$$\perp(x,y) = g^{(-1)}(g(x) + g(y)) \tag{2}$$

where

$$g^{(-1)}(x) = \begin{cases} g^{-1}(x) & \text{if } x \leq g(1) \\ 1 & \text{otherwise} \end{cases}$$

is the pseudoinverse of g. Moreover, this representation is unique up to a positive multiplicative constant.

$\perp$ *is generated by* g if $\perp$ has representation (2).

g is called *additive generator* of $\perp$.

Additive Generators of t-conorms – Two Examples

Find an additive generator g of $\bot_{\text{Łuka}}(x, y) = \min\{x + y,\ 1\}$.

for instance $g_{\text{Łuka}}(x) = x$

then, $g_{\text{Łuka}}^{(-1)}(x) = \min\{x,\ 1\}$

thus $\bot_{\text{Łuka}}(x, y) = g_{\text{Łuka}}^{(-1)}(g_{\text{Łuka}}(x) + g_{\text{Łuka}}(y))$

Find an additive generator g of $\bot_{\text{sum}}(x, y) = x + y - x \cdot y$.

to be discussed in the exercise

hint: use of logarithmic and exponential function

Now, let us examine some typical families of operations.

Sugeno-Weber Family I

For $\lambda > -1$ and $x, y \in [0, 1]$, define

$$\top_\lambda(x, y) = \max\left\{\frac{x + y - 1 + \lambda xy}{1 + \lambda},\ 0\right\},$$

$$\bot_\lambda(x, y) = \min\{x + y + \lambda xy,\ 1\}.$$

$\lambda = 0$ leads to $\top_{\text{Łuka}}$ and $\bot_{\text{Łuka}}$, resp.

$\lambda \to \infty$ results in $\top_{\text{prod}}$ and $\bot_{\text{sum}}$, resp.

$\lambda \to -1$ creates $\top_{-1}$ and $\bot_{-1}$, resp.

Sugeno-Weber Family II

Additive generators f_λ of $\top_\lambda$ are

$$f_\lambda(x) = \begin{cases} 1 - x & \text{if } \lambda = 0 \\ 1 - \frac{\log(1+\lambda x)}{\log(1+\lambda)} & \text{otherwise.} \end{cases}$$

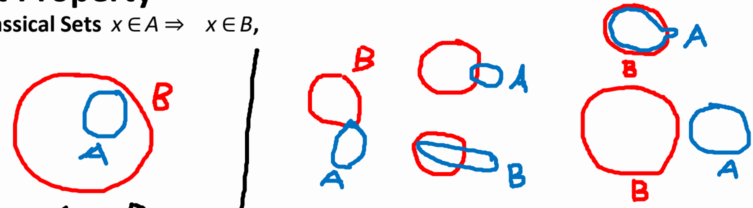
$\{\top_\lambda\}_{\lambda > -1}$ are increasing functions of parameter λ.

Additive generators of $\perp_\lambda$ are $g_\lambda(x) = 1 - f_\lambda(x)$.

Fuzzy Sets Inclusion

Subset Property

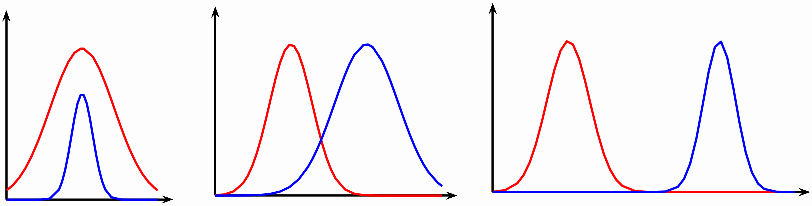
For Classical Sets $x \in A \Rightarrow \quad x \in B,$



$A \subseteq B$

not $(A \subseteq B)$

For Fuzzy Sets : $x \in \mu \Rightarrow \quad x \in \mu'$



Definition of a Fuzzy Implication

1. One way of defining I is to use the property that in classical logic the propositions a ⇒ b and $\neg a \vee b$ have the same truth values for all truth assignments to a and b.
   If we model the disjunction and negation as t-conorm and fuzzy complement, resp., then for all $a, b \in [0,1]$ the following defininion of a fuzzy implication seems reasonable:

$$I(a,b) = \perp(\sim a, b).$$

2. Another way is to use the concept of a residuum in classical logic: a ⇒ b and $\max\{x \in \{0,1\} \mid a \wedge x \leq b\}$ have the same truth values for all truth assignments for a, and b. If in a generalized logic the conjunction is modelled by a t-norm, then a reasonable generalization could be:

$$I(a,b) = \sup\{x \in [0,1] \mid \top(a,x) \leq b\}.$$

Definition of a Fuzzy Implication

3. Another proposal is to use the fact that, in classical logic, the propositions a $\Rightarrow$ b and $\neg a \vee (a \wedge b)$ have the same truth for all truth assignments.

   A possible extension to many valued logics is therefore

$$I(a, b) = \bot(\sim a, \top(a, b)),$$

   where $(\top, \bot, \sim)$ should be a *De Morgan triplet*.

So again, the classical definition of an implication is unique, whereas there is a „zoo" of fuzzy implications.

Typical question for applications: What to use when and why?

S-Implications

Implications based on $I(a,b) = \bot(\sim a, b)$ are called S-**implications**.

Symbol S is often used to denote t-conorms.

Four well-known S-implications are based on $\sim a = 1 - a$:

Name	$I(a,b)$	$\bot(a,b)$
Kleene-Dienes	$I_{\max}(a,b) = \max(1-a, b)$	$\max(a,b)$
Reichenbach	$I_{\text{sum}}(a,b) = 1 - a + ab$	$a + b - ab$
Łukasiewicz	$I_{Ł}(a,b) = \min(1,\ 1-a+b)$	$\min(1,\ a+b)$
largest	$I_{-1}(a,b) = \begin{cases} b, & \text{if } a = 1 \\ 1-a, & \text{if } b = 0 \\ 1, & \text{otherwise} \end{cases}$	$\begin{cases} b, & \text{if } a = 0 \\ a, & \text{if } b = 0 \\ 1, & \text{otherwise} \end{cases}$

R-Implications

$I(a,b) = \sup\{x \in [0,1] \mid \top(a,x) \leq b\}$ leads to R-**implications**.

Symbol R represents close connection to residuated semigroup.

Three well-known R-implications are based on $\sim a = 1 - a$:

- Standard fuzzy intersection leads to **Gödel implication**

$$I_{\min}(a,b) = \sup\{x \mid \min(a,x) \leq b\} = \begin{cases} 1, & \text{if } a \leq b \\ b, & \text{if } a > b. \end{cases}$$

- Product leads to **Goguen implication**

$$I_{\text{prod}}(a,b) = \sup\{x \mid ax \leq b\} = \begin{cases} 1, & \text{if } a \leq b \\ b/a, & \text{if } a > b. \end{cases}$$

- Łukasiewicz t-norm leads to **Łukasiewicz implication**

$$I_{\text{Ł}}(a,b) = \sup\{x \mid \max(0, a + x - 1) \leq b\} = \min(1,\ 1 - a + b).$$

QL-Implications

Implications based on $I(a,b) = \bot(\sim a, \top(a,b))$ are called *QL*-**implications** (*QL* from quantum logic).

Four well-known QL-implications are based on $\sim a = 1 - a$:

- Standard min and max lead to **Zadeh implication**

$$I_Z(a,b) = \max[1-a, \min(a,b)].$$

- The algebraic product and sum lead to

$$I_\mathrm{p}(a,b) = 1 - a + a^2 b.$$

- Using $\top_Ł$ and $\bot_Ł$ leads to **Kleene-Dienes implication** again.
- Using $\top_{-1}$ and $\bot_{-1}$ leads to

$$I_\mathrm{q}(a,b) = \begin{cases} b, & \text{if } a = 1 \\ 1-a, & \text{if } a \neq 1, b \neq 1 \\ 1, & \text{if } a \neq 1, b = 1. \end{cases}$$

All I come from generalizations of the classical implication.

They collapse to the classical implication when truth values are 0 or 1.

Generalizing classical properties leads to following propositions:

1) $a \leq b$ implies $I(a,x) \geq I(b,x)$ *(monotonicity in 1st argument)*
2) $a \leq b$ implies $I(x,a) \leq I(x,b)$ *(monotonicity in 2nd argument)*
3) $I(0,a) = 1$ *(dominance of falsity)*
4) $I(1,b) = b$ *(neutrality of truth)*
5) $I(a,a) = 1$ *(identity)*
6) $I(a,I(b,c)) = I(b,I(a,c))$ *(exchange property)*
7) $I(a,b) = 1$ if and only if $a \leq b$ *(boundary condition)*
8) $I(a,b) = I(\sim b, \sim a)$ for fuzzy complement $\sim$ *(contraposition)*
9) I is a continuous function *(continuity)*

Generator Function

I that satisfy all listed axioms are characterized by this theorem:

Theorem
A function $I : [0,1]^2 \to [0,1]$ satisfies Axioms 1–9 of fuzzy implications for a particular fuzzy complement $\sim$ if and only if there exists a strict increasing continuous function $f : [0,1] \to [0,\infty)$ such that $f(0) = 0$,

$$I(a,b) = f^{(-1)}(f(1) - f(a) + f(b))$$

for all $a, b \in [0,1]$, and

$$\sim a = f^{-1}(f(1) - f(a))$$

for all $a \in [0,1]$.

Example

Consider $f_\lambda(a) = \ln(1 + \lambda a)$ with $a \in [0, 1]$ and $\lambda > 0$.

Its pseudo-inverse is

$$f_\lambda^{(-1)}(a) = \begin{cases} \frac{e^a - 1}{\lambda}, & \text{if } 0 \le a \le \ln(1 + \lambda) \\ 1, & \text{otherwise.} \end{cases}$$

The fuzzy negation generated by f_λ for all $a \in [0, 1]$ is

$$n_\lambda(a) = \frac{1 - a}{1 + \lambda a}.$$

The resulting fuzzy implication for all $a, b \in [0, 1]$ is thus

$$I_\lambda(a, b) = \min\left(1, \frac{1 - a + b + \lambda b}{1 + \lambda a}\right).$$

If $\lambda \in (-1, 0)$, then I_λ is called **pseudo-Łukasiewicz implication**.