OTTO VON GUERICKE UNIVERSITÄT MAGDEBURG

INF FAKULTÄT FÜR INFORMATIK

Fuzzy Systems
Approximate Reasoning

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SS 2020

The Extension Principle

„Fuzzification“

How to extend $\phi : X^n \to Y$ to $\hat{\phi} : \mathcal{F}(X)^n \to \mathcal{F}(Y)$?

Let $\mu \in \mathcal{F}(\mathbb{R})$ be a fuzzy set of the imprecise concept "about 2".

Then the degree of membership $\mu(2.2)$ can be seen as *truth value* of the statement "2.2 is about equal to 2".

Let $\mu' \in \mathcal{F}(\mathbb{R})$ be a fuzzy set of the imprecise concept "old".

Then the truth value of "2.2 is about equal 2 **and** 2.2 is old" can be seen as membership degree of 2.2 *w.r.t.* imprecise concept "about 2 and old".

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Reminder: Operating with Truth Values

Any $\top$ ($\bot$) can be used to represent conjunction (disjunction).

However, now only $\top_{\min}$ and $\bot_{\max}$ shall be used.

Let P be set of imprecise statements that can be combined by *and*, *or*.

truth : $P \rightarrow [0, 1]$ assigns truth value truth(a) to every $a \in P$.

truth(a) = 0 means a is definitely false.

truth(a) = 1 means a is definitely true.

If $0 <$ truth(a) < 1, then only gradual truth of statement a.

Extension Principle

Combination of two statements $a, b \in P$:

$$\begin{aligned} \text{truth}(a \text{ and } b) = \quad & \text{truth}(a \wedge b) = \quad & \min\{\text{truth}(a), \text{truth}(b)\}, \\ \text{truth}(a \text{ or } b) = \quad & \text{truth}(a \vee b) = \quad & \max\{\text{truth}(a), \text{truth}(b)\} \end{aligned}$$

For infinite number of statements $a_i, i \in I$:

$$\begin{aligned} \text{truth}(\forall i \in I : a_i) &= \inf\left\{\text{truth}(a_i) \mid i \in I\right\}, \\ \text{truth}(\exists i \in I : a_i) &= \sup\left\{\text{truth}(a_i) \mid i \in I\right\} \end{aligned}$$

This concept helps to extend $\phi : X^n \to Y$ to $\hat{\phi} : \mathcal{F}(X)^n \to \mathcal{F}(Y)$.

- Crisp tuple $(x_1, \ldots, x_n)$ is mapped to crisp value $\phi(x_1, \ldots, x_n)$.
- Imprecise descriptions $(\mu_1, \ldots, \mu_n)$ of $(x_1, \ldots, x_n)$ are mapped to fuzzy value $\hat{\phi}(\mu_1, \ldots, \mu_n)$.

Example – How to extend the addition?

$+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \quad (a, b) \mapsto a + b$

Extensions to sets: $+ : 2^{\mathbb{R}} \times 2^{\mathbb{R}} \to 2^{\mathbb{R}}$

$$(A, B) \mapsto A + B = \{y \mid (\exists a)(\exists b)(y = a + b) \wedge (a \in A) \wedge (b \in B)\}$$

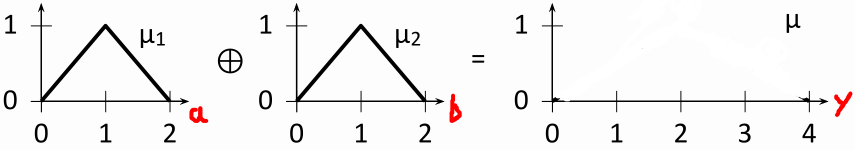
Extensions to fuzzy sets:

$$+ : \mathcal{F}(\mathbb{R}) \times \mathcal{F}(\mathbb{R}) \to \mathcal{F}(\mathbb{R}), \quad (\mu_1, \mu_2) \mapsto \mu_1 \oplus \mu_2$$

$$\begin{aligned}
\text{truth}(y \in \mu_1 \oplus \mu_2) &= \text{truth}((\exists a)(\exists b) : (y = a + b) \wedge (a \in \mu_1) \wedge (b \in \mu_2)) \\
&= \sup_{a,b}\{\text{truth}(y = a + b) \wedge \text{truth}(a \in \mu_1) \wedge \\
&\qquad \text{truth}(b \in \mu_2)\} \\
&= \sup_{a,b:y=a+b} \{\min(\mu_1(a),\ \mu_2(b))\}
\end{aligned}$$

Example – How to extend the addition?

$$\mu(y) = \sup_{a,b:y=a+b} \{\min(\mu_1(a),\ \mu_2(b))\}$$



$\mu(2) = 1$ because $\mu_1(1) = 1$ and $\mu_2(1) = 1$

$\mu(5) = 0$ because if $a + b = 5$, then $\min\{\mu_1(a), \mu_2(b)\} = 0$

$\mu(1) = 0.5$ because it is the result of an optimization task with optimum at, *e.g.* $a = 0.5$ and $b = 0.5$

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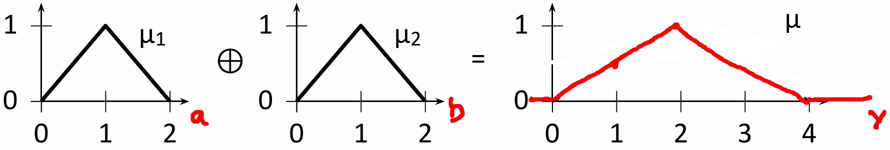

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$\mu(1) = ?$

a	b	$\mu_1(a) \wedge \mu_2(b)$
0	1	$0 \wedge 1 = 0$
1	0	$1 \wedge 0 = 0$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2} \wedge \frac{1}{2} = \frac{1}{2}$
-1	2	$0 \wedge 0 = 0$
1.5	-0.5	$\frac{1}{2} \wedge 0 = 0$
0.2	0.8	$0.2 \wedge 0.8 = 0.2$

Extension to Sets

Definition

Let $\phi : X^n \to Y$ be a mapping. The *extension* $\hat{\phi}$ of ϕ is given by

$$\hat{\phi} : [2^X]^n \to 2^Y \quad \text{with}$$

$$\hat{\phi}(A_1, \ldots, A_n) = \{y \in Y \mid \exists (x_1, \ldots, x_n) \in A_1 \times \ldots \times A_n : \phi(x_1, \ldots, x_n) = y\}.$$

Extension to Fuzzy Sets

Definition

Let $\phi : X^n \to Y$ be a mapping. The *extension* $\hat{\phi}$ of ϕ is given by

$$\hat{\phi} : [\mathcal{F}(X)]^n \to \mathcal{F}(Y) \quad \text{with}$$

$$\hat{\phi}(\mu_1, \ldots, \mu_n)(y) = \sup\{\min\{\mu_1(x_1), \ldots, \mu_n(x_n)\} \mid (x_1, \ldots, x_n) \in X^n \wedge \phi(x_1, \ldots, x_n) = y\}$$

assuming that $\sup \emptyset = 0$.

Example I

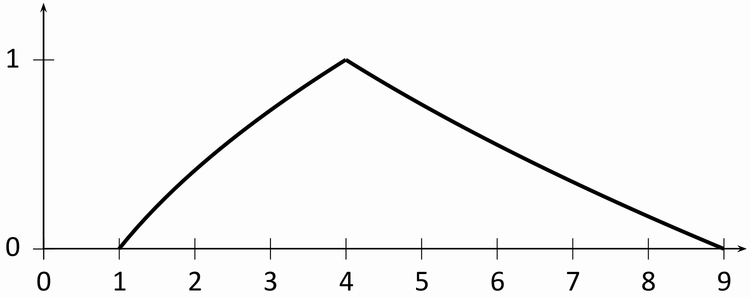
Let fuzzy set "approximately 2" be defined as

$$\mu(x) = \begin{cases} x-1, & \text{if } 1 \le x \le 2 \\ 3-x, & \text{if } 2 \le x \le 3 \\ 0, & \text{otherwise.} \end{cases}$$

The extension of $\phi : \mathbb{R} \to \mathbb{R}, x \mapsto x^2$ to fuzzy sets on $\mathbb{R}$ is

$$\begin{aligned} \hat{\phi}(\mu)(y) &= \sup\left\{\mu(x) \;\middle|\; x \in \mathbb{R} \wedge x^2 = y\right\} \\ &= \begin{cases} \sqrt{y}-1, & \text{if } 1 \le y \le 4 \\ 3-\sqrt{y}, & \text{if } 4 \le y \le 9 \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

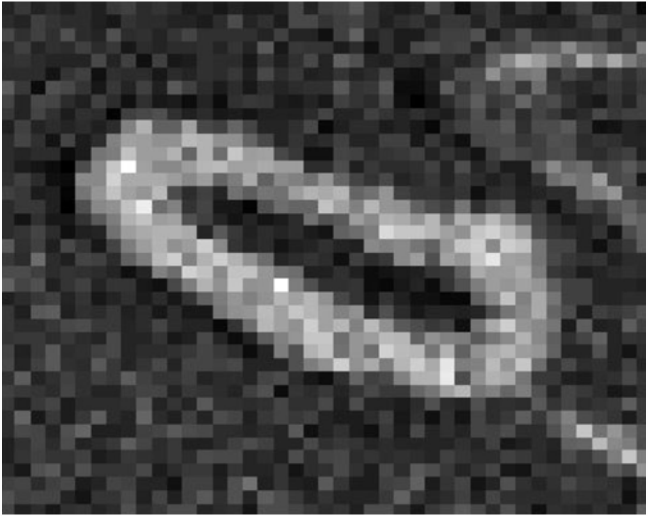
Example II



The extension principle is taken as basis for “fuzzifying” whole theories.

Fuzzy Relations

A grey level picture can be interpreted as a fuzzy set



Definition of Relation

A **relation** among crisp sets $X_1, \ldots, X_n$ is a subset of the Cartesian Product $X_1 \times \ldots \times X_n$. It is denoted as $R(X_1, \ldots, X_n)$ or $R(X_i \mid 1 \leq i \leq n)$.

So, the relation $R(X_1, \ldots, X_n) \subseteq X_1 \times \ldots \times X_n$ is set, too. The

basic concept of sets can be also applied to relations:

- containment, subset, union, intersection, complement

Each crisp relation can be defined by its characteristic function

$$R(x_1, \ldots, x_n) = \begin{cases} 1, & \text{if and only if } (x_1, \ldots, x_n) \in R, \\ 0, & \text{otherwise.} \end{cases}$$

The membership of $(x_1, \ldots, x_n)$ in R indicates whether the elements of $(x_1, \ldots, x_n)$ are related to each other or not.

Fuzzy Relations

The characteristic function of a crisp relation can be generalized to allow tuples to have degrees of membership.

A **fuzzy relation R** is a fuzzy set of $X_1 \times \ldots \times X_n$

The membership grade indicates strength of the present relation between elements of the tuple.

The fuzzy relation can also be represented by an *n*-dimensional membership array.

Example

Let R be a fuzzy relation between two sets X = {New York City, Paris} and Y = {Beijing, New York City, London}.

R shall represent relational concept "very far".

It can be represented (subjectively) as two-dimensional membership array:

	NYC	Paris
Beijing	1	0.9
NYC	0	0.7
London	0.6	0.3

Cartesian Product of Fuzzy Sets: n Dimensions

Let $A_1, \ldots, A_n$ be fuzzy sets (n ≥ 2) in $X_1, \ldots, X_n$, respectively

The (fuzzy) *Cartesian product* of $A_1, \ldots, A_n$, denoted by $A_1 \times \ldots \times A_n$, is a fuzzy relation of the product space $X_1 \times \ldots \times X_n$.

It is defined by its membership function

$$\mu_{A_1 \times \ldots \times A_n}(x_1, \ldots, x_n) = \top\,(\mu_{A_1}(x_1), \ldots, \mu_{A_n}(x_n))$$

for $x_i \in X_i,\ 1 \leq i \leq n.$

In most applications $\top = \min$ is chosen.

Cartesian Product of Fuzzy Sets in two Dimensions

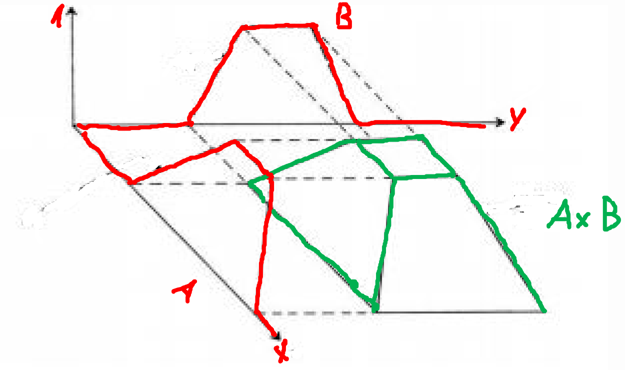
A special case of the Cartesian product is when $n = 2$.

Then the Cartesian product of fuzzy sets $A \in \mathcal{F}(X)$ and $B \in \mathcal{F}(Y)$ is a fuzzy relation $A \times B \in \mathcal{F}(X \times Y)$ defined by

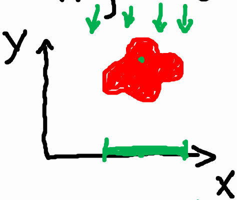
$$\mu_{A\times B}(x, y) = \top[\mu_A(x),\ \mu_B(y)],\ \text{for all } x \in X, \text{ and } y \in Y.$$

Example:

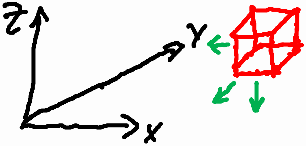
Cartesian Product in $F(X \times Y)$ with t-norm = min



Projection

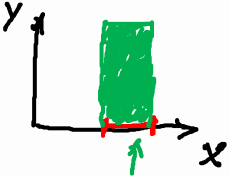


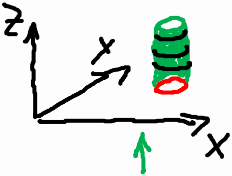
2 projections



6 projections

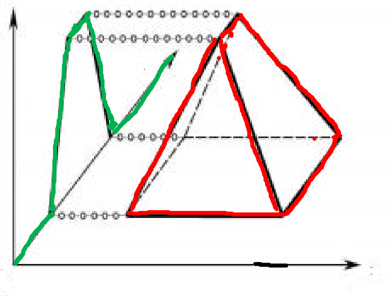
Cylindrical Extension

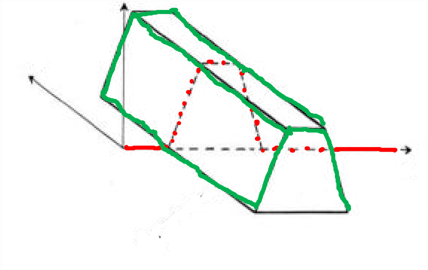




projection of μ

cylindrical extension of μ





Example

Consider the sets $X_1 = \{0, 1\}$, $X_2 = \{0, 1\}$, $X_3 = \{0, 1, 2\}$ and the ternary fuzzy relation on $X_1 \times X_2 \times X_3$ defined as follows:

Let $R_{ij} = [R \downarrow \{X_i, X_j\}]$ and $R_i = [R \downarrow \{X_i\}]$ for all $i, j \in \{1, 2, 3\}$. Using this notation, all possible projections of R are given below.

$(x_1,$	$x_2,$	$x_3)$	$R(x_1, x_2, x_3)$	$R_{12}(x_1, x_2)$	$R_{13}(x_1, x_3)$	$R_{23}(x_2, x_3)$	$R_1(x_1)$	$R_2(x_2)$	$R_3(x_3)$
0	0	0	0.4	0.9	1.0	0.5	1.0	0.9	1.0
0	0	1	0.9	0.9	0.9	0.9	1.0	0.9	0.9
0	0	2	0.2	0.9	0.8	0.2	1.0	0.9	1.0
0	1	0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
0	1	1	0.0	1.0	0.9	0.5	1.0	1.0	0.9
0	1	2	0.8	1.0	0.8	1.0	1.0	1.0	1.0
1	0	0	0.5	0.5	0.5	0.5	1.0	0.9	1.0
1	0	1	0.3	0.5	0.5	0.9	1.0	0.9	0.9
1	0	2	0.1	0.5	1.0	0.2	1.0	0.9	1.0
1	1	0	0.0	1.0	0.5	1.0	1.0	1.0	1.0
1	1	1	0.5	1.0	0.5	0.5	1.0	1.0	0.9
1	1	2	1.0	1.0	1.0	1.0	1.0	1.0	1.0

Example: Detailed Calculation

Here, only consider the projection R_{12}:

$(x_1,$	$x_2,$	$x_3)$	$R(x_1, x_2, x_3)$	$R_{12}(x_1, x_2)$
0	0	0	0.4	
0	0	1	0.9	$(0,0) \mapsto \max[R(0,0,0), R(0,0,1), R(0,0,2)] = 0.9$
0	0	2	0.2	
0	1	0	1.0	
0	1	1	0.0	$(0,1) \mapsto \max[R(0,1,0), R(0,1,1), R(0,1,2)] = 1.0$
0	1	2	0.8	
1	0	0	0.5	
1	0	1	0.3	$(1,0) \mapsto \max[R(1,0,0), R(1,0,1), R(1,0,2)] = 0.5$
1	0	2	0.1	
1	1	0	0.0	
1	1	1	0.5	$(1,1) \mapsto \max[R(1,1,0), R(1,1,1), R(1,1,2)] = 1.0$
1	1	2	1.0	

Binary Fuzzy Relations

Representation and Inverse

Consider *e.g.* the **membership matrix** $\boldsymbol{R} = [r_{xy}]$ with $r_{xy} = R(x,y)$.

Its **inverse** $R^{-1}(Y,X)$ of $R(X,Y)$ is a relation on $Y \times X$ defined by

$$R^{-1}(y,x) = R(x,y) \quad \text{for all } x \in X,\ y \in Y.$$

$\boldsymbol{R}^{-1} = [r_{xy}^{-1}]$ representing $R^{-1}(y,x)$ is the transpose of $\boldsymbol{R}$ for $R(X,Y)$

$$(\boldsymbol{R}^{-1})^{-1} = \boldsymbol{R}$$

Standard Composition

Consider the binary relations $P(X,Y)$, $Q(Y,Z)$ with common set Y.

The **standard composition** of P and Q is defined as

$$(x,z) \in P \circ Q \iff \exists y \in Y : \{(x,y) \in P \wedge (y,z) \in Q\}.$$

In the fuzzy case this is generalized by

$$[P \circ Q](x,z) = \sup_{y \in Y} \{\min\{P(x,y),\ Q(y,z)\}\},\ \text{for all } x \in X \quad and\ z \in Z.$$

If Y is finite, sup operator can be replaced by max.

The standard composition is also called **max-min composition**.

Inverse of Standard Composition

The inverse of the max-min composition follows from its definition:

$$[P(X,Y) \circ Q(Y,Z)]^{-1} = Q^{-1}(Z,Y) \circ P^{-1}(Y,X).$$

Its associativity also comes directly from its definition:

$$[P(X,Y)] \circ Q(Y,Z)] \circ R(Z,W) = P(X,Y) \circ [Q(Y,Z) \circ R(Z,W)].$$

Note that the standard composition is not commutative.

Matrix notation: $[r_{ij}] = [p_{ik}] \circ [q_{kj}]$ with $r_{ij} = \max_k \min(p_{ik}, q_{kj})$.

Example

$$P \circ Q = R$$

$$\begin{bmatrix} .3 & .5 & .8 \\ 0 & .7 & 1 \\ .4 & .6 & .5 \end{bmatrix} \circ \begin{bmatrix} .9 & .5 & .7 & .7 \\ .3 & .2 & 0 & .9 \\ 1 & 0 & .5 & .5 \end{bmatrix} = \begin{bmatrix} .8 & .3 & .5 & .5 \\ 1 & .2 & 5 & .7 \\ .5 & .4 & .5 & .5 \end{bmatrix}$$

For instance:

$$\begin{aligned} r_{11} &= \max\{\min(p_{11}, q_{11}), \min(p_{12}, q_{21}), \min(p_{13}, q_{31})\} \\ &= \max\{\min(.3, .9), \min(.5, .3), \min(.8, 1)\} \\ &= .8 \\ r_{32} &= \max\{\min(p_{31}, q_{12}), \min(p_{32}, q_{22}), \min(p_{33}, q_{32})\} \\ &= \max\{\min(.4, .5), \min(.6, .2), \min(.5, 0)\} \\ &= .4 \end{aligned}$$

Example: Properties of Airplanes (Speed, Height, Type)

4 possible speeds:	s_1, s_2, s_3, s_4
3 heights:	h_1, h_2, h_3
2 types:	t_1, t_2

Consider the following fuzzy relations for airplanes:

- relation A between maximal speed and maximal height,
- relation B between maximal height and the type.

$\boldsymbol{A}$	h_1	h_2	h_3
s_1	1	.2	0
s_2	.1	1	0
s_3	0	1	1
s_4	0	.3	1

$\boldsymbol{B}$	t_1	t_2
h_1	1	0
h_2	.9	1
h_3	0	.9

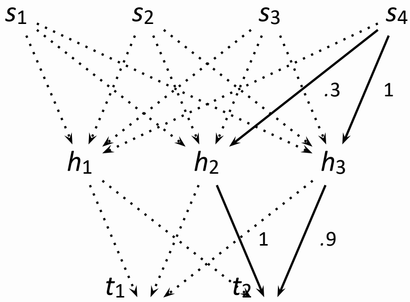
Example (cont.)

matrix multiplication scheme

			1	0
A	$\circ$	**B**	.9	1
			0	.9
1	.2	0	1	.2
.1	1	0	.9	1
0	1	1	.9	1
0	.3	1	.3	.9

$A \circ B$ speed-type relation

flow scheme



$$(A \circ B)(s_4, t_2) = \max\{\min\{.3, 1\}, \min\{1, .9\}\}$$
$$= .9$$

Binary Relations on a Single Set

Binary Relations on a Single Set

It is also possible to define crisp or fuzzy binary relations among elements of a single set X.

Such a binary relation can be denoted by $R(X,X)$ or $R(X^2)$ which is a subset of $X \times X = X^2$.

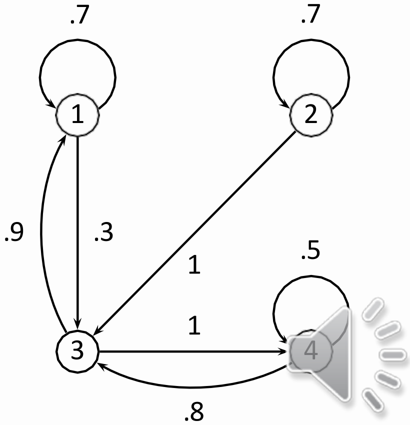
These relations are often referred to as **directed graphs** which is also a representation of them.

- Each element of X is represented as node.
- Directed connections between nodes indicate pairs of $x \in X$ for which the grade of the membership is nonzero.
- Each connection is labeled by its actual membership grade of the corresponding pair in R.

Example

An example of $R(X,X)$ defined on $X = \{1,2,3,4\}$. Two different representation are shown below.

	1	2	3	4
1	.7	0	.3	0
2	0	.7	1	0
3	.9	0	0	1
4	0	0	.8	.5



Fuzzy Arithmetic

Fuzzy Sets of $\mathbb{R}$

There are many different types of fuzzy sets $\mu : \mathbb{R} \to [0, 1]$,

They play important role in many applications, *e.g.* fuzzy control, decision making, approximate reasoning, optimization, and statistics with imprecise probabilities.

Some Special Fuzzy Sets I

Here, we only consider special classes $\mathcal{F}(\mathbb{R})$ of fuzzy sets μ on $\mathbb{R}$.

Definition

(a) $\mathcal{F}_N(\mathbb{R}) \stackrel{\text{def}}{=} \{\mu \in \mathcal{F}(\mathbb{R}) \mid \exists x \in \mathbb{R} : \mu(x) = 1\}$,

(b) $\mathcal{F}_C(\mathbb{R}) \stackrel{\text{def}}{=} \{\mu \in \mathcal{F}_N(\mathbb{R}) \mid \forall \alpha \in (0, 1] : [\mu]_\alpha \text{ is compact }\}$,

(c) $\mathcal{F}_I(\mathbb{R}) \stackrel{\text{def}}{=} \{\mu \in \mathcal{F}_N(\mathbb{R}) \mid \forall a, b, c \in \mathbb{R} : c \in [a, b] \Rightarrow \mu(c) \geq \min\{\mu(a), \mu(b)\}\}$.

Some Special Fuzzy Sets II

An element in F_N (IR) is called **normal fuzzy set**:

- It's meaningful if $\mu \in F_N$ (IR) is used as *imprecise description* of an existing (but not precisely defined) set $\subseteq$ IR.
- In such cases it would not be plausible to assign maximum membership degree of 1 to no single real number at all.

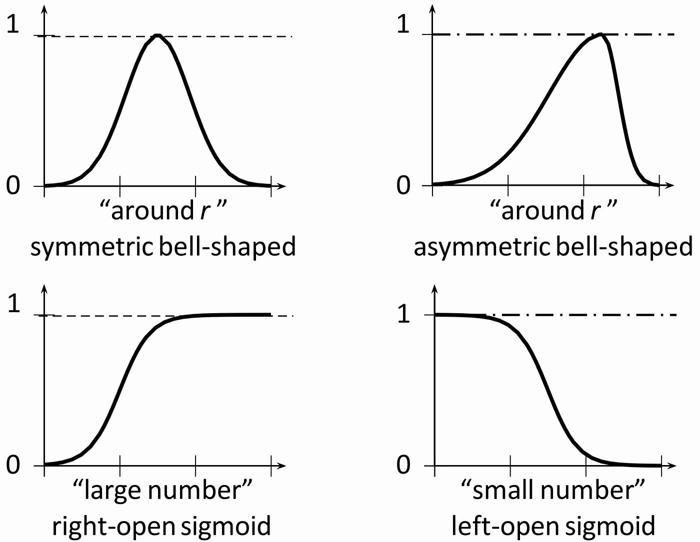
Sets in F_C (IR) are **uppersemi-continuous**:

- Function f is upper semi-continuous at point x_0 iff $\limsup_{x \to x_0} f(x) \leq f(x_0)$.
- This property simplifies arithmetic operations.

Fuzzy sets in F_I (IR) are called **fuzzy intervals**:

- The are *normal* and *fuzzy convex*.
- Their core is a classical interval.
- If $\mu \in F_I$ (IR) is used for describing an imprecise real number, then often people say: **μ is a fuzzy number.**

Basic Types of Fuzzy Sets of $\mathbb{R}$



Quantitative Fuzzy Variables

The concept of a fuzzy number plays fundamental role in formulating *quantitative fuzzy variables*.

These are variables whose states are fuzzy numbers.

When the fuzzy numbers represent linguistic concepts, *e.g.*

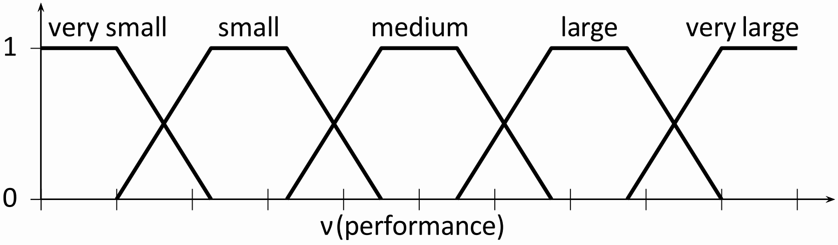
very small, *small*, *medium*, etc.

then final constructs are called **linguistic variables**.

Each linguistic variable is defined in terms of *base variable* which is a variable in classical sense, *e.g.* temperature, pressure, age.

Linguistic terms representing approximate values of base variable are captured by appropriate fuzzy numbers.

Linguistic Variable



Each linguistic variable is defined by quintuple (v, T, X, g, m).

- *name* v of the variable
- set T of *linguistic terms* of v
- [illegible] $X \subseteq \mathbb{R}$
- *syntactic rule* g (grammar) for generating linguistic terms
- *semantic rule* m that assigns *meaning* $m(t)$ to every $t \in T$, *i.e.* $m : T \rightarrow F(X)$

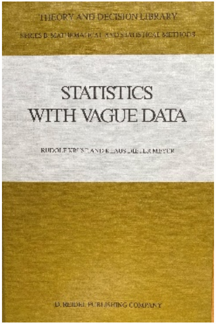
Operations on Linguistic Variables

To deal with linguistic variables, consider

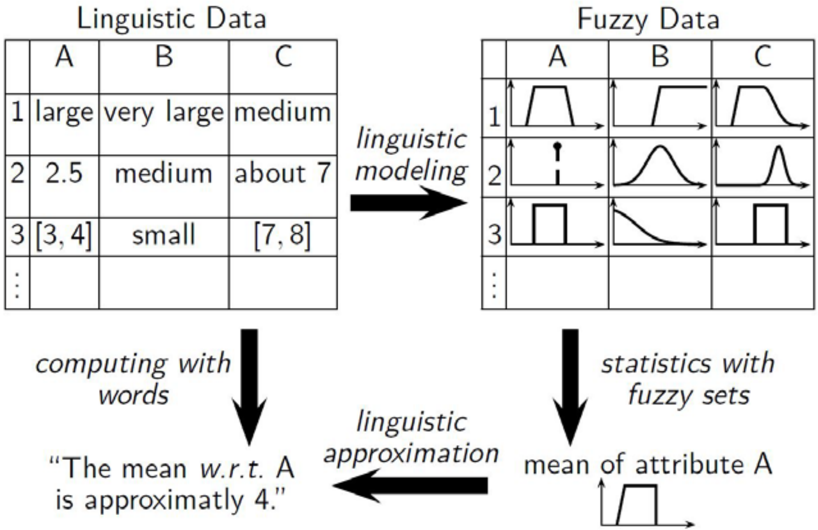
- not only set-theoretic operations
- but also arithmetic operations on fuzzy numbers (*i.e.* interval arithmetic).

Statistics with vague data

- Given a sample = (*small*, *medium*, *small*, *large*, ...).
- How to define mean value or standard deviation?



Analysis of Linguistic Data



Example – Application of Linguistic Data

Consider the problem to model the climatic conditions of several towns.

A tourist may want information about tourist attractions.

Assume that linguistic random samples are based on subjective observations of selected people, *e.g.*

- climatic attribute *clouding*
- linguistic values *cloudless, clear, fair, cloudy, ...*

Example – Linguistic Modeling by an Expert

The attribute *clouding* is modeled by elementary linguistic values, *e.g*

$$
\begin{aligned}
\text{cloudless} &\mapsto \text{sigmoid}(0, -0.07) \\
\text{clear} &\mapsto \text{Gauss}(25, 15) \\
\text{fair} &\mapsto \text{Gauss}(50, 20) \\
\text{cloudy} &\mapsto \text{Gauss}(75, 15) \\
\text{overcast} &\mapsto \text{sigmoid}(100, 0.07) \\
\text{exactly}(x) &\mapsto \text{exact}(x) \\
\text{approx}(x) &\mapsto \text{Gauss}(x, 3) \\
\text{between}(x, y) &\mapsto \text{rectangle}(x, y) \\
\text{approx\_between}(x, y) &\mapsto \text{trapezoid}(x - 20, x, y, y + 20)
\end{aligned}
$$

where $x, y \in [0, 100] \subseteq \mathbb{R}$.

Example

Gauss(a, b) is, *e.g.* a function defined by

$$f(x) = \exp\left(-\left(\frac{x-a}{b}\right)^2\right), \qquad x, a, b \in \mathbb{R}, \quad b > 0$$

induced language of expressions:

$$\begin{aligned} \langle\text{expression}\rangle \; := \; & \langle\text{elementary linguistic value}\rangle \mid \\ & (\ \langle\text{expression}\rangle\ ) \mid \\ & \{\ \text{not} \mid \text{dil} \mid \text{con} \mid \text{int}\ \}\ \langle\text{expression}\rangle \mid \\ & \langle\text{expression}\rangle\ \{\ \text{and} \mid \text{or}\ \}\ \langle\text{expression}\rangle\ , \end{aligned}$$

e.g. approx(x) *and cloudy* is represented by function

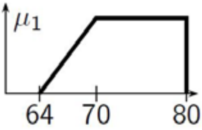
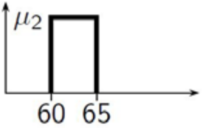
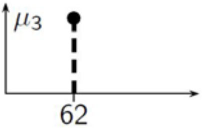
$$\min\{\text{Gauss}(x, 3), \text{Gauss}(75, 15)\}\ .$$

Example – Linguistic Random Sample

Attribute	:	Clouding
Observations	:	Limassol, Cyprus
2009/10/23	:	cloudy
2009/10/24	:	dil approx_between(50, 70)
2009/10/25	:	fair or cloudy
2009/10/26	:	approx(75)
2009/10/27	:	dil(clear or fair)
2009/10/28	:	int cloudy
2009/10/29	:	con fair
2009/11/30	:	approx(0)
2009/11/31	:	cloudless
2009/11/01	:	cloudless or dil clear
2009/11/02	:	overcast
2009/11/03	:	cloudy and between(70, 80)
. ..	:	. . .
2009/11/10	:	clear

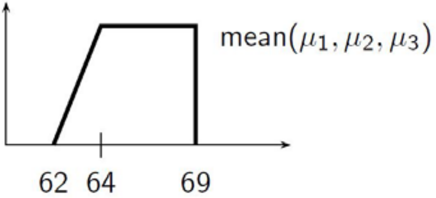
Statistics with fuzzy sets are necessary to analyze linguistic data.

Example – Ling. Random Sample of 3 People

no.	age (linguistic data)	age (fuzzy data)
1	approx. between 70 and 80 and definitely not older than 80	
2	between 60 and 65	
3	62	

Example – Mean Value of Ling. Random Sample

$$\text{mean}(\mu_1, \mu_2, \mu_3) = \frac{1}{3}(\mu_1 \oplus \mu_2 \oplus \mu_3)$$



i.e. approximately between 64 and 69 but not older than 69

Efficient Operations I

How to define arithmetic operations for calculating with $\mathcal{F}(\mathbb{R})$?

Using extension principle for sum $\mu \oplus \mu'$, product $\mu \odot \mu'$ and reciprocal value $\operatorname{rec}(\mu)$ of arbitrary fuzzy sets $\mu, \mu' \in \mathcal{F}(\mathbb{R})$

$$(\mu \oplus \mu')(t) = \sup \{\min\{\mu(x_1), \mu'(x_2)\} \mid x_1, x_2 \in \mathbb{R}, x_1 + x_2 = t\},$$
$$(\mu \odot \mu')(t) = \sup \{\min\{\mu(x_1), \mu'(x_2)\} \mid x_1, x_2 \in \mathbb{R}, x_1 \cdot x_2 = t\},$$
$$\operatorname{rec}(\mu)(t) = \sup \left\{\mu(x) \;\middle|\; x \in \mathbb{R} \setminus \{0\}, \frac{1}{x} = t\right\}.$$

In general, operations on fuzzy sets are much more complicated (especially if vertical instead of horizontal representation is applied).

It's desirable to reduce fuzzy arithmetic to ordinary set arithmetic.

Then, we apply elementary operations of *interval arithmetic*.

Efficient Operations II

Definition
A family $(A_\alpha)_{\alpha \in (0,1)}$ of sets is called *set representation* of $\mu \in \mathcal{F}_N(\mathbb{R})$ if

(a) $0 < \alpha < \beta < 1 \Longrightarrow A_\beta \subseteq A_\alpha \subseteq \mathbb{R}$ and

(b) $\mu(t) = \sup \{\alpha \in [0,1] \mid t \in A_\alpha\}$

holds where $\sup \emptyset = 0$.

Theorem
Let $\mu \in F_N(\mathbb{R})$. The family $(A_\alpha)_{\alpha \in (0,1)}$ of sets is a set representation of μ if and only if

$$[\mu]_{\underline{\alpha}} = \{t \in \mathbb{R} \mid \mu(t) > \alpha\} \subseteq A_\alpha \subseteq \{t \in \mathbb{R} \mid \mu(t) \geq \alpha\} = [\mu]_\alpha$$

is valid for all $\alpha \in (0,1)$.

Efficient Operations III

Theorem

Let $\mu_1, \mu_2, \ldots, \mu_n$ be normal fuzzy sets of $\mathbb{R}$ and $\phi : \mathbb{R}^n \to \mathbb{R}$ be a mapping. Then the following holds:

(a) $\forall \alpha \in [0,1) : [\hat{\phi}(\mu_1, \ldots, \mu_n)]_{\underline{\alpha}} = \phi([\mu_1]_{\underline{\alpha}}, \ldots, [\mu_n]_{\underline{\alpha}})$,

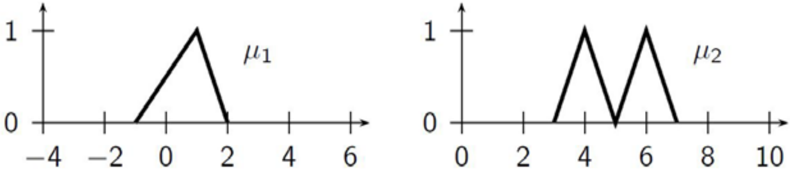
(b) $\forall \alpha \in (0,1] : [\hat{\phi}(\mu_1, \ldots, \mu_n)]_{\alpha} \supseteq \phi([\mu_1]_{\alpha}, \ldots, [\mu_n]_{\alpha})$,

(c) if $((A_i)_\alpha)_{\alpha \in (0,1)}$ *is a set representation of* μ_i *for* $1 \leq i \leq n$*, then* $(\phi((A_1)_\alpha, \ldots, (A_n)_\alpha))_{\alpha \in (0,1)}$ *is a set representation of* $\hat{\phi}(\mu_1, \ldots, \mu_n)$.

For arbitrary mapping ϕ, set representation of its extension $\hat{\phi}$ can be obtained with help of set representation $((A_i)_\alpha)_{\alpha \in (0,1)}, i = 1, 2, \ldots, n$.

It's used to carry out arithmetic operations on fuzzy sets efficiently.

Example I



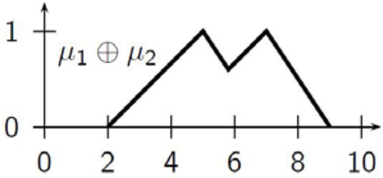
For μ_1, μ_2, the set representations are

- $[\mu_1]_\alpha = [2\alpha - 1, 2 - \alpha]$,
- $[\mu_2]_\alpha = [\alpha + 3, 5 - \alpha] \cup [\alpha + 5, 7 - \alpha]$.

Let $\mathrm{add}(x, y) = x + y$, then $(A_\alpha)_{\alpha \in (0,1)}$ represents $\mu_1 \oplus \mu_2$

$$\begin{aligned} A_\alpha &= \mathrm{add}([\mu_1]_\alpha, [\mu_2]_\alpha) = [3\alpha + 2, 7 - 2\alpha] \cup [3\alpha + 4, 9 - 2\alpha] \\ &= \begin{cases} [3\alpha + 2, 7 - 2\alpha] \cup [3\alpha + 4, 9 - 2\alpha], & \text{if } \alpha \geq 0.6 \\ [3\alpha + 2, 9 - 2\alpha], & \text{if } \alpha \leq 0.6. \end{cases} \end{aligned}$$

Example II



$$(\mu_1 \oplus \mu_2)(x) = \begin{cases} \frac{x-2}{3}, & \text{if } 2 \le x \le 5 \\ \frac{7-x}{2}, & \text{if } 5 \le x \le 5.8 \\ \frac{x-4}{3}, & \text{if } 5.8 \le x \le 7 \\ \frac{9-x}{2}, & \text{if } 7 \le x \le 9 \\ 0, & \text{otherwise} \end{cases}$$

Interval Arithmetic

Determining the set representations of arbitrary combinations of fuzzy sets can be reduced very often to simple interval arithmetic.

Using fundamental operations of arithmetic leads to the following ($a, b, c, d \in \mathbb{R}$):

$$[a, b] + [c, d] = [a + c, b + d]$$
$$[a, b] - [c, d] = [a - d, b - c]$$
$$[a, b] \cdot [c, d] = \begin{cases} [ac, bd], & \text{for } a \geq 0 \wedge c \geq 0 \\ [bd, ac], & \text{for } b < 0 \wedge d < 0 \\ [\min\{ad, bc\}, \max\{ad, bc\}], & \text{for } ab \geq 0 \wedge cd \geq 0 \wedge ac < 0 \\ [\min\{ad, bc\}, \max\{ac, bd\}], & \text{for } ab < 0 \vee cd < 0 \end{cases}$$
$$\frac{1}{[a, b]} = \begin{cases} \left[\frac{1}{b}, \frac{1}{a}\right], & \text{if } 0 \notin [a, b] \\ \left[\frac{1}{b}, \infty\right) \cup \left(-\infty, \frac{1}{a}\right], & \text{if } a < 0 \wedge b > 0 \\ \left[\frac{1}{b}, \infty\right), & \text{if } a = 0 \wedge b > 0 \\ \left(-\infty, \frac{1}{a}\right], & \text{if } a < 0 \wedge b = 0 \end{cases}$$

In general, set representation of α-cuts of extensions $\hat{\phi}(\mu_1, \ldots, \mu_n)$ cannot be determined directly from α-cuts.

It works always for continuous ϕ and fuzzy sets in $\mathcal{F}_C(\mathbb{R})$.

Theorem

Let $\mu_1, \mu_2, \ldots, \mu_n \in \mathcal{F}_C(\mathbb{R})$ *and* $\phi : \mathbb{R}^n \to \mathbb{R}$ *be a continuous mapping. Then*

$$\forall \alpha \in (0, 1] : [\hat{\phi}(\mu_1, \ldots, \mu_n)]_\alpha = \phi([\mu_1]_\alpha, \ldots, [\mu_n]_\alpha).$$

The horizontal representation is often useful.
Finding $\hat{\phi}$ values is easier than directly applying the extension principle.
However, all α-cuts cannot be stored in a computer.
Only a finite number of α-cuts can be stored.

Fuzzy Control Basics

Fuzzy Control

Biggest success of fuzzy systems in industry and commerce.

Special kind of non-linear table-based control method.

Definition of non-linear transition function can be made without specifying each entry individually.

Examples: technical systems

- Electrical engine moving an elevator,
- Heating installation

Goal: define certain behavior

- Engine should maintain certain number of revolutions per minute.
- Heating should guarantee certain room temperature.

Table-based Control

Control systems all share a time-dependent **output variable**:

- Revolutions per minute,
- Room temperature.

Output is controlled by **control variable**:

- Adjustment of current,
- Thermostat.

Also, **disturbance variables** influence output:

- Load of elevator, ...,
- Outside temperature or sunshine through a window, ...

Table-based Control

Computation of actual value incorporates both

control variable measurements of current output variable ξ and

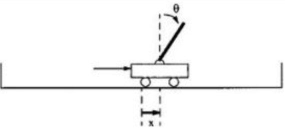
change of output variable $\Delta\xi = \frac{d\xi}{dt}$.

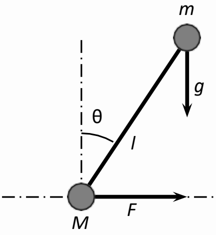
If ξ is given in finite time intervals,

then set $\Delta\xi(t_{n+1}) = \xi(t_{n+1}) - \xi(t_n)$.

In this case measurement of $\Delta\xi$ not necessary.

Example: Cartpole Problem





Balance an upright standing pole

Lower end of pole can be moved unrestrained along horizontal axis.

Mass *m* at foot and mass *M* at head.

Influence of mass of shaft itself is negligible.

Determine force *F* (control variable) that is necessary to balance pole standing upright.

That is measurement of following output variables:

- angle θ of pole in relation to vertical axis,
- change of angle, *i.e.* triangular velocity $\dot{\theta} = \frac{d\theta}{dt}$

Both should converge to zero.

Notation

Input variables $\xi_1, \dots, \xi_n$, control variable η

Measurements: used to determine actual value of η

Assumption: ξ_i, $1 \leq i \leq n$ is value of X_i, $\eta \in Y$

Solution: **control function** φ

$$\begin{aligned} \varphi : X_1 \times \ldots \times X_n &\to Y \\ (x_1, \dots, x_n) &\mapsto y \end{aligned}$$

Example: Cartpole Problem (cont.)

Angle $\theta \in X_1 = [-90\,°, 90\,°]$

Theoretically, every angle velocity $\dot{\theta}$ possible.

Extreme $\dot{\theta}$ are artificially achievable.

Assume $-45\,°/\mathrm{s} \leq \dot{\theta} \leq 45\,°/\mathrm{s}$ holds,
i.e. $\dot{\theta} \in X_2 = [-45\,°/\mathrm{s}, 45\,°/\mathrm{s}]$.

Absolute value of force $|F| \leq 10\,\mathrm{N}$.

Thus define $F \in Y = [-10\,\mathrm{N}, 10\,\mathrm{N}]$.

Example: Cartpole Problem (cont.)

Differential equation of cartpole problem:

$$(M+m)\sin^2\theta \cdot l \cdot \ddot{\theta} + m \cdot l \cdot \sin\theta\cos\theta \cdot \dot{\theta}^2 - (M+m)\cdot g \cdot \sin\theta = -F \cdot \cos\theta$$

Compute $F(t)$ such that $\theta(t)$ and $\dot{\theta}(t)$ converge towards zero quickly.

Physical analysis demands knowledge about physical process.

Problems of Classical Approach

Often very difficult or even impossible to specify accurate mathematical model.

Description with differential equations is very complex.

Profound physical knowledge from engineer.

Exact solution can be very difficult.

Should be possible: to control process without physical-mathematical model,
e.g. human being knows how to ride bike without knowing existence of differential equations.

Fuzzy Approach

Simulate behavior of human who knows how to control.

That is a **knowledge-based analysis**.

Directly ask expert to perform analysis.

Then expert specifies knowledge as **linguistic rules**, *e.g.* for cartpole problem:

"If θ is approximately zero and $\dot{\theta}$ is also approximately zero, then F has to be approximately zero, too."

Fuzzy Approach: Fuzzy Partitioning

1. Formulate set of linguistic rules:

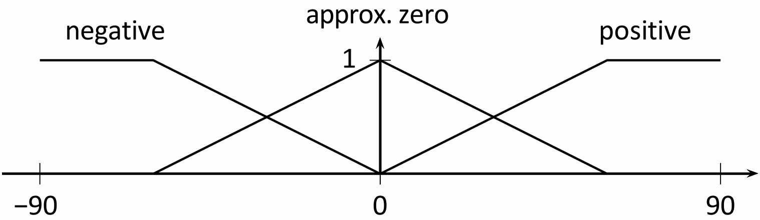
Determine linguistic terms (represented by fuzzy sets).

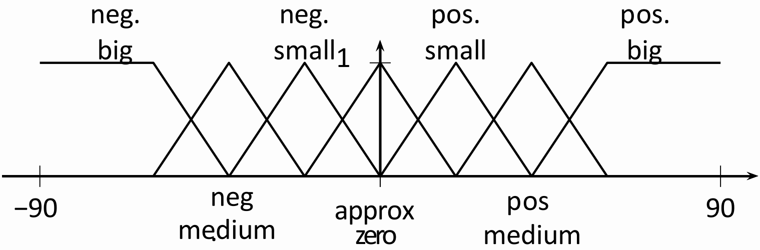
$X_1, \ldots, X_n$ and Y is partitioned into fuzzy sets.

Define p_1 distinct fuzzy sets $\mu_1^{(1)}, \ldots, \mu_{p_1}^{(1)} \in \mathcal{F}(X_1)$ on set X_1.

Associate linguistic term with each set.

Coarse and Fine Fuzzy Partitions





Fuzzy Approach: Fuzzy Partitioning II

X_1 corresponds to interval $[a, b]$ of real line,
$\mu_1^{(1)}, \dots, \mu_{p_1}^{(1)} \in \mathcal{F}(X_1)$ are triangular functions

$$\begin{aligned}\mu_{x_0,\varepsilon} : [a, b] &\to [0, 1]\\ x &\mapsto 1 - \min\{\varepsilon \cdot |x - x_0|,\ 1\}.\end{aligned}$$

If $a < x_1 < \dots < x_{p_1} < b$, only $\mu_2^{(1)}, \dots, \mu_{p_1-1}^{(1)}$ are triangular.

Boundaries are treated differently.

Fuzzy Approach: Fuzzy Partitioning III

left fuzzy set:

$$\mu_1^{(1)} : [a, b] \to [0, 1]$$

$$x \mapsto \begin{cases} 1, & \text{if } x \leq x_1 \\ 1 - \min\{\varepsilon \cdot (x - x_1),\ 1\} & \text{otherwise} \end{cases}$$

right fuzzy set:

$$\mu_{p_1}^{(1)} : [a, b] \to [0, 1]$$

$$x \mapsto \begin{cases} 1, & \text{if } x_{p_1} \leq x \\ 1 - \min\{\varepsilon \cdot (x_{p_1} - x),\ 1\} & \text{otherwise} \end{cases}$$

Example: Cartpole Problem (cont.)

X_1 partitioned into 7 fuzzy sets.

Similar fuzzy partitions for X_2 and Y.

Next step: specify rules

$$\text{if } \xi_1 \text{ is } A^{(1)} \text{ and } \dots \text{ and } \xi_n \text{ is } A^{(n)} \text{ then } \eta \text{ is } B,$$

$A^{(1)}, \dots, A^{(n)}$ and B represent linguistic terms corresponding to $\mu^{(1)}, \dots, \mu^{(n)}$ and μ according to $X_1, \dots, X_n$ and Y.

Rule base consists of k rules.

Example: Cartpole Problem (cont.)

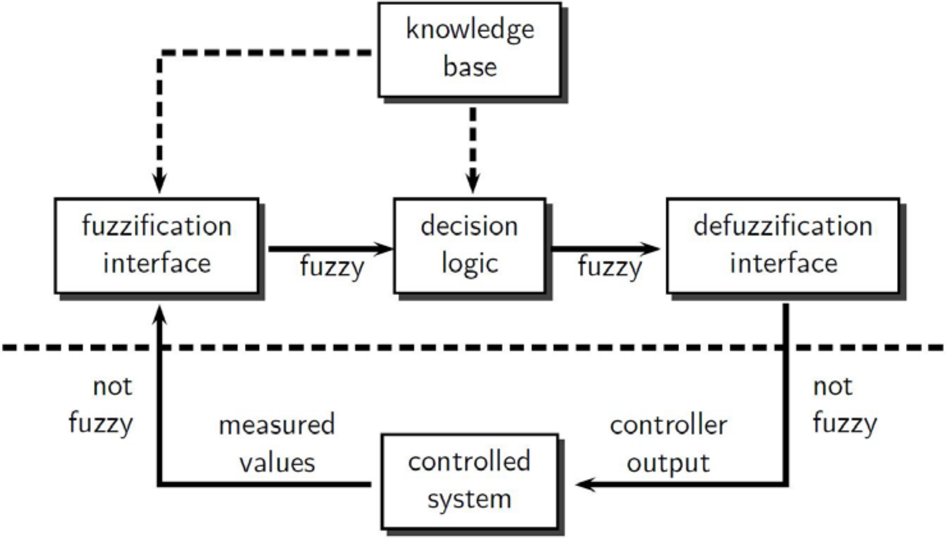
θ

		nb	nm	ns	az	ps	pm	pb
	nb			ps	pb			
	nm				pm			
	ns	nm		ns	ps			
$\dot{\theta}$	az	nb	nm	ns	az	ps	pm	pb
	ps				ns	ps		pm
	pm				nm			
	pb				nb	ns		

19 rules for cartpole problem, often not necessary to determine all table entries *e.g.*

If θ is *approximately zero* and $\dot{\theta}$ is *negative medium* then *F* is *positive medium*.

Architecture of a Fuzzy Controller



Fuzzy Approach: Challenge

Develop the desired table-based controller by using (imprecise and fuzzy) knowledge, i.e. fuzzy rules.

Main Problem:

How to define function $\varphi : X_1 \times X_2 \rightarrow Y$ that fits to a fuzzy rule set?

Fuzzy Rule Bases

Approximate Reasoning with Fuzzy Rules

General schema

Rule 1:	**if** X is M_1, **then** Y is N_1
Rule 2:	**if** X is M_2, **then** Y is N_2
:	:
Rule r :	**if** X is M_r , **then** Y is N_r
Fact:	X is M'
Conclusion:	Y is N'

Given *r* **if-then rules** and fact “X is M'”, we conclude “Y is N'”.

Expert like to describe a function f by using fuzzy rules.

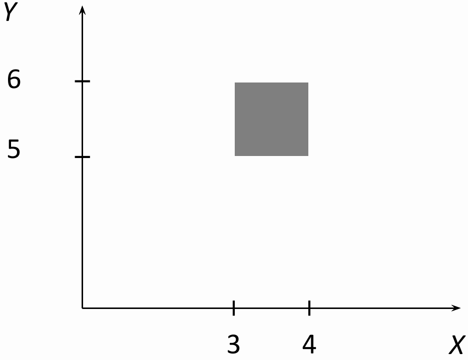
This modus ponens style of reasoning is often used in **fuzzy controllers.**

What is the exact meaning of an imprecise rule and how to use it?

Method 1

Imprecise rule: **if** $X = [3, 4]$ **then** $Y = [5, 6]$.

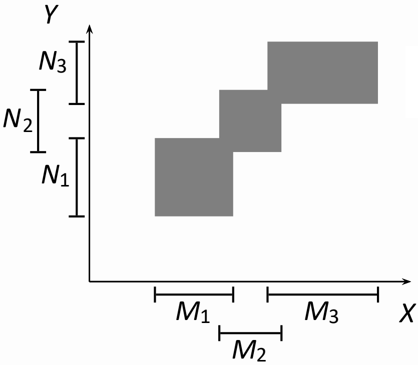
Interpretation : $[3, 4] \times [5, 6]$ is a „patch“, where the function „passes“.



Method 1: Several Rules

Several imprecise rules: **if** $X = M_1$ **then** $Y = N_1$, **if** $X = M_2$ **then** $Y = N_2$, **if** $X = M_3$ **then** $Y = N_3$.

Disjunctive Interpretation: Several rules form a "patchwork rug" for the function's graph.

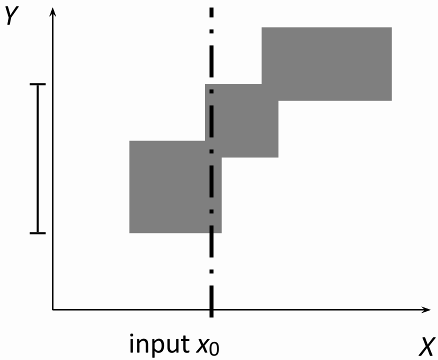


$$S = \bigcup_{i=1}^{r} M_i \times N_i$$

Method 1: Conclusion

Possible output

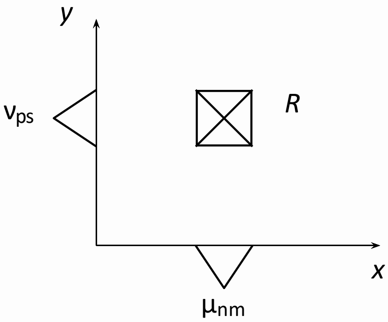
$B = \{x_0\} \circ S = B$



Method 1: Fuzzy Rules

one fuzzy rule:

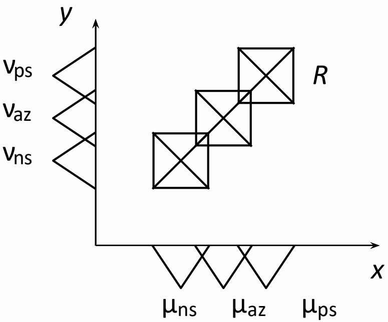
if X = nm **then** Y = ps



$R = \mu_{nm} \times \nu_{ps}$

several fuzzy rules:

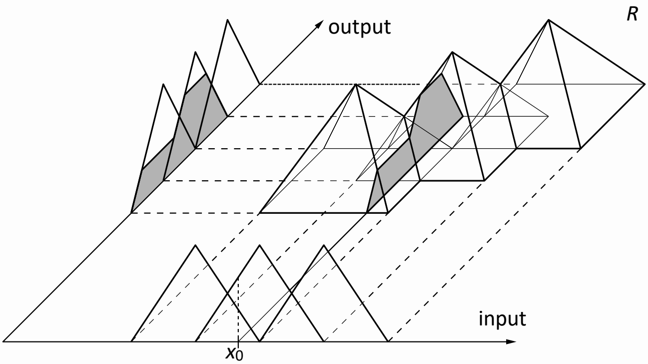
ns → ns', az → az', ps → ps'



$R = \mu_{ns} \times \nu_{ns'} \cup$

$\mu_{az} \times \nu_{az'} \cup \mu_{ps} \times \nu_{ps'}$

Method 1: Fuzzy Conclusion



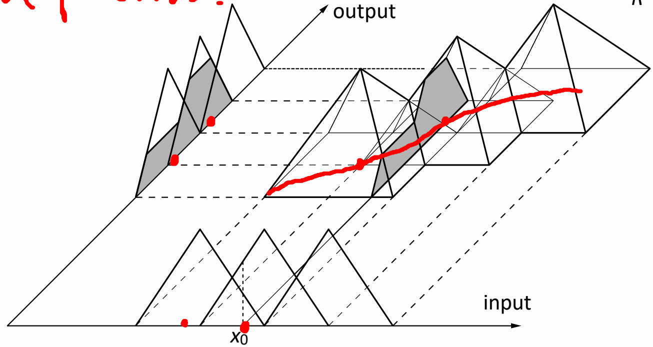
Three fuzzy rules

Every pyramid is specified by a fuzzy rule (Cartesian product).

Input x_0 leads to gray-shaded fuzzy output $\{x_0\} \circ R$.

Method 1: Fuzzy Rules for Interpolation

Control function?



Three fuzzy rules

Every pyramid is specified by a fuzzy rule (Cartesian product).

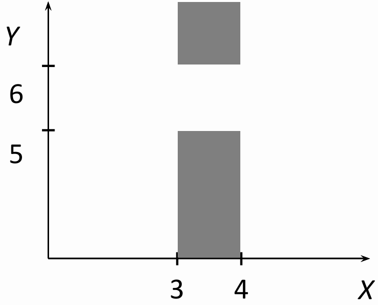
Input x_0 leads to gray-shaded fuzzy output $\{x_0\} \circ R$.

Method 2

Imprecise rule: **if** $X = [3, 4]$ **then** $Y = [5, 6]$

Interpretation: Constraints

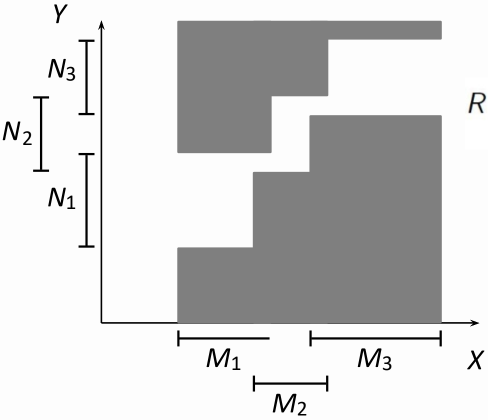
Black values are impossible, white ones are allowed.



Method 2: Rules are constraints

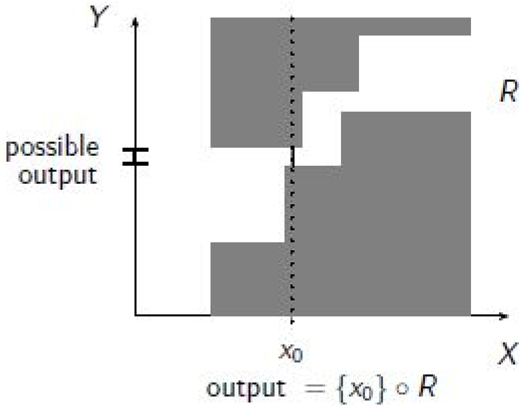
Several imprecise rules: **if** $X = M_1$ **then** $Y = N_1$, **if** $X = M_2$ **then** $Y = N_2$, **if** $X = M_3$ **then** $Y = N_3$

Conjunctive interpretation: "corridor" (white area) for the function



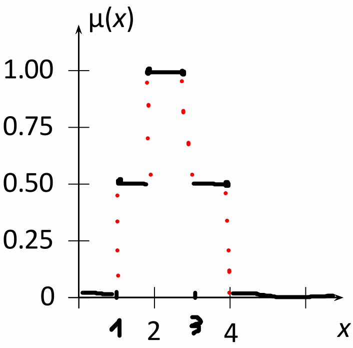
$$R = \bigcap_{i=1}^{r} (M_i \times N_i) \cup (M_i^C \times Y)$$

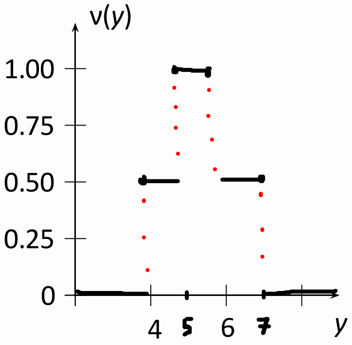
Method 2: Conclusion in case of imprecise rules



Method 2: Fuzzy Rules

if *X* is approx. 2.5 **then** *Y* is approx.5.5

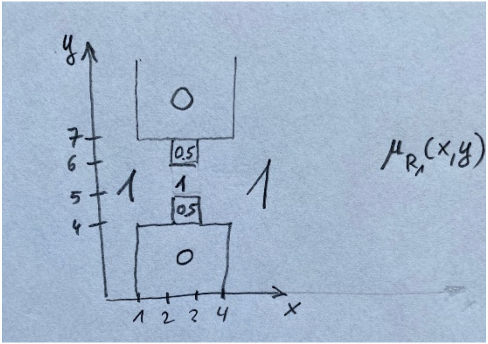




Method 2: Fuzzy Rule modelled as Fuzzy Relation by Gödel Implication

$$R_1 : \textbf{if } X = \mu_{M_1} \textbf{ then } Y = \nu_{B_1}$$

$$\mu_{R_1} : X \times Y \to [0,1], \quad (x,y) \longrightarrow \begin{cases} 1 & \text{if } \mu_{M_1}(x) \le \nu_{B_1}(y), \\ \nu_{B_1}(y) & \text{otherwise.} \end{cases}$$



Method 2 : Conjunctive Fuzzy Rule Bases

R_1 : **if** $X = \mu_{M_1}$ **then** $Y = \nu_{B_1}$, ..., R_n : **if** $X = \mu_{M_n}$ **then** $Y = \nu_{B_n}$

$$\mu_R = \min_{1 \le i \le r} \mu_{R_i}$$

For input μ_A, the output is η ,

$$\eta(y) = \sup_{x \in X} \min\{\mu_A(x), \mu_R(x,y)\}.$$

Method 3

Fuzzy Relational Equations

Given $\mu_1, \ldots, \mu_r$ of X and $\nu_1, \ldots, \nu_r$ of Y and r rules *if* μ_i *then* ν_i.

What is a **fuzzy relation** ϱ that fits the rule system?

One solution is to find a relation ϱ such that

$$\forall i \in \{1, \ldots, r\} : \nu_i = \mu_i \circ \varrho,$$

$$\mu \circ \varrho : Y \to [0, 1], \quad y \mapsto \sup_{x \in X} \min\{\mu(x),\ \varrho(x, y)\}.$$

Example

$$\mu_A = \begin{pmatrix} .9 & 1 & .7 \end{pmatrix} \qquad \nu_B = \begin{pmatrix} 1 & .4 & .8 & .7 \end{pmatrix}$$

$$\varrho_{A \widetilde{\bigcirc} B} = \begin{pmatrix} 1 & .4 & .8 & .7 \\ 1 & .4 & .8 & .7 \\ 1 & .4 & 1 & 1 \end{pmatrix} \qquad \varrho_1 = \begin{pmatrix} 0 & 0 & 0 & .7 \\ 1 & .4 & .8 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{array}{ccc|cccc} & & & 1 & .4 & .8 & .7 \\ & & & 1 & .4 & .8 & .7 \\ & & & 1 & .4 & 1 & 1 \\ \hline .9 & 1 & .7 & 1 & .4 & .8 & .7 \end{array} \qquad \varrho_2 = \begin{pmatrix} 0 & .4 & .8 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & .7 \end{pmatrix}$$

$\varrho_{A \widetilde{\bigcirc} B}$ largest solution, ϱ_1, ϱ_2 are two minimal solutions.

Solution space forms upper semilattice.

Solution of a Relational Equation

Theorem

i) *Let "if A then B" be a rule with $\mu_A \in \mathcal{F}(X)$ and $\nu_B \in \mathcal{F}(Y)$. Then the relational equation $\nu_B = \mu_A \circ \varrho$ can be solved iff the Gödel relation $\varrho_{A \circledS B}$ is a solution.*
$\varrho_{A \circledS B} : X \times Y \to [0,1]$ is defined by

$$(x, y) \mapsto \begin{cases} 1 & \text{if } \mu_A(x) \leq \nu_B(y), \\ \nu_B(y) & \text{otherwise.} \end{cases}$$

ii) *If ϱ is a solution, then the set of solutions $R = \{\varrho_S \in \mathcal{F}(X \times Y) \mid \nu_B = \mu_A \circ \varrho_S\}$ has the following property: If $\varrho_{S'}, \varrho_{S''} \in R$, then $\varrho_{S'} \cup \varrho_{S''} \in R$.*

iii) *If $\varrho_{A \circledS B}$ is a solution, then $\varrho_{A \circledS B}$ is the largest solution w.r.t. $\subseteq$.*

Solution of a Set of Relational Equations

Generalization of this result to system of r relational equations:

Theorem

Let $\nu_{B_i} = \mu_{A_i} \circ \varrho$ for $i = 1, \ldots, r$ be a system of relational equations.

i) There is a solution iff $\bigcap_{i=1}^{r} \varrho_{A_i \circledS B_i}$ is a solution.

ii) If $\bigcap_{i=1}^{r} \varrho_{A_i \circledS B_i}$ is a solution, then this solution is the biggest solution w.r.t. $\subseteq$.

Remark: if there is no solution, then Gödel relation is often at least a good approximation.