OTTO VON GUERICKE UNIVERSITÄT MAGDEBURG

INF FAKULTÄT FÜR INFORMATIK

Fuzzy Systems

Fuzzy Sets and Fuzzy Logic

Prof. Dr. Rudolf Kruse **Alexander Dockhorn**

{kruse,dockhorn}@ovgu.de

Otto-von-Guericke University of Magdeburg

Faculty of Computer Science

Institute of Intelligent Cooperating Systems

Motivation

Motivation

Every day humans use imprecise linguistic terms
e.g. big, *fast*, *about 12 o'clock*, *old*, etc.

All complex human actions are decisions based on such concepts:

- driving and parking a car,
- financial/business decisions,
- law and justice,
- giving a lecture,
- listening to the professor/tutor.

So, these terms and the way they are processed play a crucial role.

Computers need a mathematical model to express and process such complex semantics.

Concepts of classical mathematics are inadequate for such models.

Lotfi Asker Zadeh (1965)

Classes of objects in the real world do not have precisely defined criteria of membership.

Such imprecisely defined "classes" play an important role in human thinking,

Particularly in domains of pattern recognition, communication of information, and abstraction.

Zadeh in 2004 (born 1921)

Example – The Sorites Paradox

If a sand dune is small, adding one grain of sand to it leaves it small.
A sand dune with a single grain is small.

Hence all sand dunes are small.

Paradox comes from all-or-nothing treatment of *small*.

Degree of truth of “heap of sand is small” decreases by adding one grain after another.

Certain number of words refer to continuous numerical scales.

Example – The Sorites Paradox

How many grains of sand has a sand dune at least?

Statement $A(n)$: "n grains of sand are a sand dune."

Let $d_n = T(A(n))$ denote "degree of acceptance" for $A(n)$.

Then

$$0 = d_0 \leq d_1 \leq \ldots \leq d_n \leq \ldots \leq 1$$

can be seen as truth values of a **many valued logic**.

Imprecision

Consider the notion *bald*:
A man without hair on his head is bald,
a hairy man is not bald.

Usually, *bald* is only partly applicable.

Where to set *baldness/non baldness* threshold?

Fuzzy set theory does not assume any threshold!

Lotfi A. Zadeh's Principle of Incompatibility

> *"Stated informally, the essence of this principle is that as the complexity of a system increases, our ability to make precise and yet significant statements about its behavior diminishes until a threshold is reached beyond which precision and significance (or relevance) become almost mutually exclusive characteristics."*

Fuzzy sets/fuzzy logic are used as mechanism for abstraction of unnecessary or too complex details.

Applications of Fuzzy Systems

Control Engineering: Idle Speed Control for VW Beetle

Approximate Reasoning: Fuzzy Rule Based Systems

Data Analysis:

- Fuzzy Clustering
- Statistics with Imprecise data
- Neuro-Fuzzy Systems

Rudolf Kruse received IEEE Fuzzy Pioneer Award for „Learning Methods for Fuzzy Systems“in 2018

Washing Machines Use Fuzzy Logic

Source: `http://www.siemens-home.com/`

Fuzzy Sets

Membership Functions

Lotfi A. Zadeh (1965)
"A fuzzy set is a class with a continuum of membership grades."

An imprecisely defined set M can often be characterized by a *membership function* μ_M.

μ_M associates real number in $[0, 1]$ with each element $x \in X$.

Value of μ_M at x represents *grade of membership* of x in M.

A Fuzzy set is defined as mapping

$$\mu : X \mapsto [0, 1].$$

Fuzzy sets μ_M generalize the notion of a characteristic function

$$\chi_M : X \mapsto \{0, 1\}.$$

Membership Functions

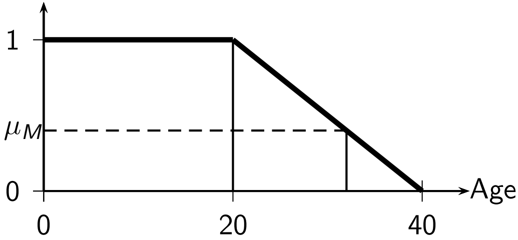
$\mu_M(u) = 1$ reflects full membership in M.

$\mu_M(u) = 0$ expresses absolute non-membership in M.

Sets can be viewed as special case of fuzzy sets where only full membership and absolute non-membership are allowed.

Such sets are called *crisp sets* or Boolean sets.

Membership degrees $0 < \mu_M < 1$ represent *partial membership*.



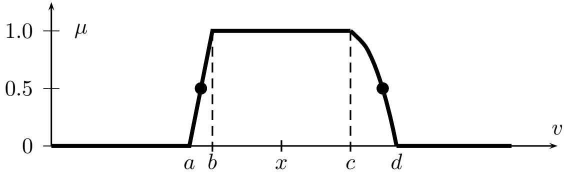
Representing *young* in "a young person"

Membership Functions

A Membership function attached to a given linguistic description (such as *young*) depends on context:
A young retired person is certainly older than young student.
Even idea of young student depends on the user.

Membership degrees are fixed only *by convention*:
Unit interval as range of membership grades is arbitrary.
Natural for modeling membership grades of fuzzy sets of real numbers.

Example – Velocity of Rotating Hard Disk



Fuzzy set μ characterizing velocity of rotating hard disk.

Let x be velocity v of rotating hard disk in revolutions per minute.

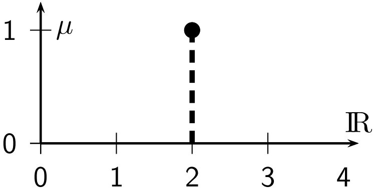
If no observations about x available, use expert's knowledge:
"It's *impossible* that v drops under a or exceeds d.
"It's highly certain that any value between $[b, c]$ can occur."

Additionally, values of v with membership degree of 0.5 are provided.

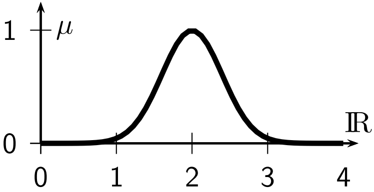
Interval $[a, d]$ is called *support* of the fuzzy set.

Interval $[b, c]$ is denoted as *core* of the fuzzy set.

Examples for Fuzzy Numbers



exactly two



around two

Exact numerical value has membership degree of 1.

Left: monotonically increasing, right: monotonically decreasing, *i.e.* unimodal function.

Terms like *around* modeled using triangular or Gaussian function.

Representation of Fuzzy Sets

Definition of a “set”

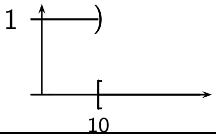
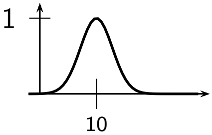
“By a set we understand every collection made into a whole of definite, distinct objects of our intuition or of our thought.” (Georg Cantor).

Georg Cantor (1845-1918)

For a set in Cantor’s sense, the following properties hold:

- $x \neq \{x\}$.
- If $x \in X$ and $X \in Y$, then $x \notin Y$.
- The Set of all subsets of X is denoted as 2^X.
- $\emptyset$ is the empty set and thus very important.

Extension to a fuzzy set

ling. description		model	
all numbers smaller than 10	→ *objective*		characteristic function of a set
all numbers <u>almost</u> equal to 10	→ *subjective*		membership function of a "fuzzy set"

Definition

A fuzzy set μ of $X \neq \emptyset$ is a function from the reference set X to the unit interval, *i.e.* $\mu : X \to [0, 1]$. $\mathcal{F}(X)$ represents the set of all fuzzy sets of X, *i.e.* $\mathcal{F}(X) \stackrel{\text{def}}{=} \{\mu \mid \mu : X \to [0, 1]\}$.

Vertical Representation

So far, fuzzy sets were described by
their characteristic/membership function and
assigning degree of membership $\mu(x)$ to each element $x \in X$.

That is the **vertical representation** of the corresponding fuzzy set, *e.g.* linguistic expression like “about m”

$$\mu_{m,d}(x) = \begin{cases} 1 - \left|\frac{m-x}{d}\right|, & \text{if } m-d \leq x \leq m+d \\ 0, & \text{otherwise,} \end{cases}$$

or “approximately between b and c”

$$\mu_{a,b,c,d}(x) = \begin{cases} \frac{x-a}{b-a}, & \text{if } a \leq x < b \\ 1, & \text{if } b \leq x \leq c \\ \frac{x-d}{c-d}, & \text{if } c < x \leq d \\ 0, & \text{if } x < a \text{ or } x > d. \end{cases}$$

Horizontal Representation

Another representation is very often applied as follows:

For all membership degrees α belonging to chosen subset of $[0, 1]$, human expert lists elements of X that fulfill vague concept of fuzzy set with degree $\geq \alpha$.

That is the **horizontal representation** of fuzzy sets by their α-**cuts**.

Definition
Let $\mu \in \mathcal{F}(X)$ and $\alpha \in [0, 1]$. Then the sets

$$[\mu]_\alpha = \{x \in X \mid \mu(x) \geq \alpha\}, \quad [\mu]_{\underline{\alpha}} = \{x \in X \mid \mu(x) > \alpha\}$$

are called the α-*cut* and *strict* α-*cut* of μ.

A Simple Example

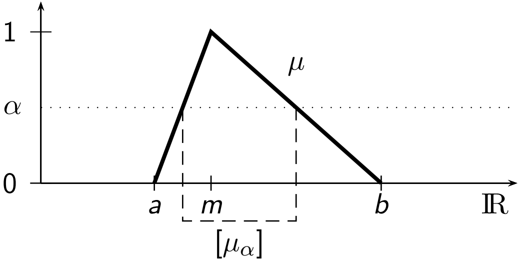
Let $A \subseteq X, \chi_A : X \to [0,1]$

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{otherwise} \end{cases}$$

Then $[\chi_A]_\alpha = A$ for $0 < \alpha \leq 1$.

χ_A is called indicator function or characteristic function of A.

An Example



Let μ be triangular function on $\mathbb{R}$ as shown above.

α-cut of μ can be constructed by

1. drawing horizontal line parallel to x-axis through point $(0, \alpha)$,
2. projecting this section onto x-axis.

$$[\mu]_\alpha = \begin{cases} [a + \alpha(m - a), b - \alpha(b - m)], & \text{if } 0 < \alpha \leq 1, \\ \mathbb{R}, & \text{if } \alpha = 0. \end{cases}$$

Properties of α-cuts I

Any fuzzy set can be described by specifying its α-cuts.

That is the α-cuts are important for application of fuzzy sets.

Theorem
Let $\mu \in \mathcal{F}(X)$, $\alpha \in [0,1]$ *and* $\beta \in [0,1]$.

(a) $[\mu]_0 = X$,

(b) $\alpha < \beta \Longrightarrow [\mu]_\alpha \supseteq [\mu]_\beta$,

(c) $\bigcap_{\alpha:\alpha<\beta} [\mu]_\alpha = [\mu]_\beta$.

Properties of α-cuts II

Theorem (Representation Theorem)
Let $\mu \in \mathcal{F}(X)$. Then

$$\mu(x) = \sup_{\alpha \in [0,1]} \left\{ \min(\alpha, \chi_{[\mu]_\alpha}(x)) \right\}$$

where $\chi_{[\mu]_\alpha}(x) = \begin{cases} 1, & \text{if } x \in [\mu]_\alpha \\ 0, & \text{otherwise.} \end{cases}$

So, fuzzy set can be obtained as upper envelope of its α-cuts.

Simply draw α-cuts parallel to horizontal axis in height of α.

In applications it is recommended to select finite subset $L \subseteq [0, 1]$ of relevant degrees of membership.

They must be semantically distinguishable.

That is, fix level sets of fuzzy sets to characterize only for these levels.

System of Sets

In this manner we obtain **system of sets**

$$\mathcal{A} = (A_\alpha)_{\alpha \in L}, \quad L \subseteq [0,1], \quad \text{card}(L) \in \mathbb{N}.$$

$\mathcal{A}$ must satisfy consistency conditions for $\alpha, \beta \in L$:

(a) $0 \in L \Longrightarrow A_0 = X$, (fixing of reference set)

(b) $\alpha < \beta \Longrightarrow A_\alpha \supseteq A_\beta$. (monotonicity)

This induces fuzzy set

$$\mu_{\mathcal{A}} : X \to [0,1],$$
$$\mu_{\mathcal{A}}(x) = \sup_{\alpha \in L} \{\min(\alpha, \chi_{A_\alpha}(x))\}.$$

If L is not finite but comprises all values $[0,1]$, then μ must satisfy

(c) $\bigcap_{\alpha:\alpha<\beta} A_\alpha = A_\beta$. (condition for continuity)

Representation of Fuzzy Sets

Definition
$\mathcal{F}L(X)$ denotes the set of all families $(A_\alpha)_{\alpha \in [0,1]}$ of sets that satisfy

(a) $A_0 = X$,

(b) $\alpha < \beta \Longrightarrow A_\alpha \supseteq A_\beta$,

(c) $\bigcap\limits_{\alpha:\alpha<\beta} A_\alpha = A_\beta$.

Any family $\mathcal{A} = (A_\alpha)_{\alpha \in [0,1]}$ of sets of X that satisfy (a)–(b) represents fuzzy set $\mu_\mathcal{A} \in \mathcal{F}(X)$ with

$$\mu_\mathcal{A}(x) = \sup \{\alpha \in [0,1] \mid x \in A_\alpha\}.$$

Vice versa: If there is $\mu \in \mathcal{F}(X)$,
then family $([\mu]_\alpha)_{\alpha \in [0,1]}$ of α-cuts of μ satisfies (a)–(b).

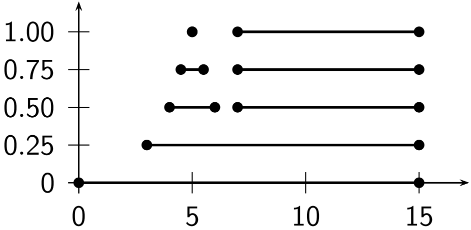
"Approximately 5 or greater than or equal to 7"

An Exemplary Horizontal View

Suppose that $X = [0, 15]$.

An expert chooses $L = \{0, 0.25, 0.5, 0.75, 1\}$ and α-cuts:

- $A_0 \quad = [0, 15]$,
- $A_{0.25} = [3, 15]$,
- $A_{0.5} \; = [4, 6] \cup [7, 15]$,
- $A_{0.75} = [4.5, 5.5] \cup [7, 15]$,
- $A_1 \quad = \{5\} \cup [7, 15]$.



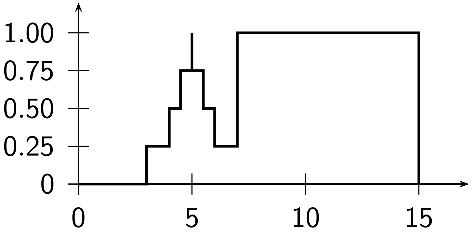
The family $(A_\alpha)_{\alpha \in L}$ of sets induces upper shown fuzzy set.

"Approximately 5 or greater than or equal to 7"

An Exemplary Vertical View

$\mu_{\mathcal{A}}$ is obtained as upper envelope of the family $\mathcal{A}$ of sets.

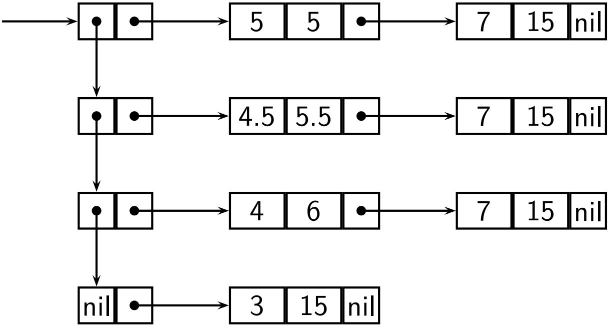
The difference between horizontal and vertical view is obvious:



The horizontal representation is easier to process in computers.

Also, restricting the domain of x-axis to a discrete set is usually done.

Horizontal Representation in the Computer



Fuzzy sets are usually stored as chain of linear lists.

For each α-level, $\alpha \neq 0$.

A finite union of closed intervals is stored by their bounds.

This data structure is appropriate for arithmetic operators.

Support and Core of a Fuzzy Set

Definition
The *support* $S(\mu)$ of a fuzzy set $\mu \in \mathcal{F}(X)$ is the crisp set that contains all elements of X that have nonzero membership. Formally

$$S(\mu) = [\mu]_{\underline{0}} = \{x \in X \mid \mu(x) > 0\}\,.$$

Definition
The *core* $C(\mu)$ of a fuzzy set $\mu \in \mathcal{F}(X)$ is the crisp set that contains all elements of X that have membership of one. Formally,

$$C(\mu) = [\mu]_1 = \{x \in X \mid \mu(x) = 1\}\,.$$

Height of a Fuzzy Set

Definition
The *height* $h(\mu)$ of a fuzzy set $\mu \in \mathcal{F}(X)$ is the largest membership grade obtained by any element in that set. Formally,

$$h(\mu) = \sup_{x \in X} \{\mu(x)\}.$$

$h(\mu)$ may also be viewed as supremum of α for which $[\mu]_\alpha \neq \emptyset$.

Definition
A fuzzy set μ is called *normal, iff* $h(\mu) = 1$.
It is called *subnormal, iff* $h(\mu) < 1$.

Convex Fuzzy Sets

Definition
Let X be a vector space. A fuzzy set $\mu \in \mathcal{F}(X)$ is called *fuzzy convex* if its α-cuts are convex for all $\alpha \in (0, 1]$.

The membership function of a convex fuzzy set **is not a** convex function.

The classical definition: The membership functions are actually **concave**.

Fuzzy Numbers

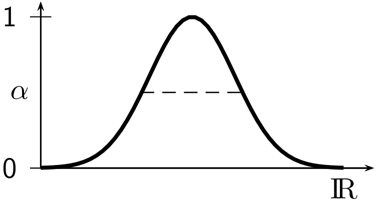
Definition
μ is a fuzzy number if and only if μ is normal and $[\mu]_\alpha$ is bounded, closed, and convex $\forall \alpha \in (0, 1]$.

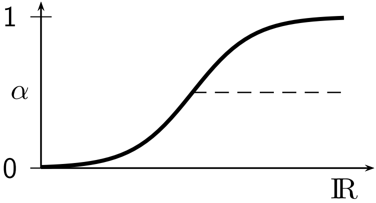
Example:

The term *approximately* x_0 is often described by a parametrized class of membership functions, *e.g.*

$$\mu_1(x) = \max\{0,\ 1 - c_1|x - x_0|\}, \qquad c_1 > 0,$$
$$\mu_2(x) = \exp(-c_2\|x - x_0\|_p), \qquad c_2 > 0, \quad p \geq 1.$$

Convex Fuzzy Sets





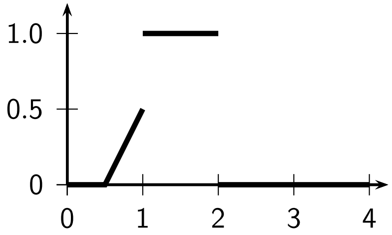
Theorem

A fuzzy set $\mu \in \mathcal{F}(\mathbb{R})$ *is convex if and only if*

$$\mu(\lambda x_1 + (1 - \lambda)x_2) \geq \min\{\mu(x_1), \mu(x_2)\}$$

for all $x_1, x_2 \in \mathbb{R}$ *and all* $\lambda \in [0, 1]$.

Fuzzy Numbers – Example



$$[\mu]_\alpha = \begin{cases} [1,2] & \text{if } \alpha \geq 0.5, \\ [0.5+\alpha, 2) & \text{if } 0 < \alpha < 0.5, \\ \mathbb{R} & \text{if } \alpha = 0 \end{cases}$$

Upper semi-continuous functions are often convenient in applications.

In many applications (*e.g.* fuzzy control) the class of the functions and their exact parameters have a limited influence on the results.

Only local monotonicity of the functions is really necessary.

In other applications (*e.g.* medical diagnosis) more precise membership degrees are needed.

Multi-valued Logics

Set Operators...

...are defined by using traditional logics operator

Let X be universe of discourse (universal set):

$$A \cap B = \{x \in X \mid x \in A \wedge x \in B\}$$
$$A \cup B = \{x \in X \mid x \in A \vee x \in B\}$$
$$A^c = \{x \in X \mid x \notin A\} = \{x \in X \mid \neg(x \in A)\}$$

$$A \subseteq B \text{ if and only if } (x \in A) \rightarrow (x \in B) \text{ for all } x \in X$$

One idea to define fuzzy set operators: use fuzzy logics.

The Traditional or Aristotlelian Logic

What is logic about? Different schools speak different languages!

There are raditional, linguistic, psychological, epistemological and mathematical schools.

Traditional logic has been founded by Aristotle (384-322 B.C.).

Aristotlelian logic can be seen as formal approach to human reasoning.

It's still used today in Artificial Intelligence for knowledge representation and reasoning about knowledge.

Detail of "The School of Athens" by R. Sanzio (1509) showing Plato (left) and his student Aristotle (right).

Classical Logic: An Overview

Logic studies methods/principles of **reasoning**.

Classical logic deals with **propositions** (either *true* or *false*).

The *propositional logic* handles combination of *logical variables*.

Key idea: how to express *n*-ary logic functions with **logic primitives**, *e.g.* $\neg, \wedge, \vee, \rightarrow$.

A set of logic primitives is **complete** if any logic function can be composed by a finite number of these primitives,
e.g. $\{\neg, \wedge, \vee\}$, $\{\neg, \wedge\}$, $\{\neg, \rightarrow\}$, $\{\downarrow\}$ (NOR), $\{|\}$ (NAND)
(this was also discussed during the 1st exercise).

Inference Rules

When a variable represented by logical formula is:
true for all possible truth values, *i.e.* it is called **tautology**,
false for all possible truth values, *i.e.* it is called **contradiction**.

Various forms of tautologies exist to perform **deductive inference**

They are called **inference rules**:

$$(a \wedge (a \rightarrow b)) \rightarrow b \qquad \textit{(modus ponens)}$$

$$(\neg b \wedge (a \rightarrow b)) \rightarrow \neg a \qquad \textit{(modus tollens)}$$

$$((a \rightarrow b) \wedge (b \rightarrow c)) \rightarrow (a \rightarrow c) \qquad \textit{(hypothetical syllogism)}$$

e.g. modus ponens: given two true propositions a and $a \rightarrow b$ (*premises*), truth of proposition b (*conclusion*) can be inferred.

Every tautology remains a tautology when any of its variables is replaced with an arbitrary logic formula.

Boolean Algebra

The propositional logic based on finite set of logic variables is isomorphic to **finite set theory**.

Both of these systems are isomorphic to a finite **Boolean algebra**.

Definition
A *Boolean algebra* on a set B is defined as quadruple $\mathcal{B} = (B, +, \cdot, ^{-})$ where B has at least two elements (bounds) 0 and 1, $+$ and $\cdot$ are binary operators on B, and $^{-}$ is a unary operator on B for which the following properties hold.

Properties of Boolean Algebras I

(B1)	Idempotence	$a + a = a$	$a \cdot a = a$
(B2)	Commutativity	$a + b = b + a$	$a \cdot b = b \cdot a$
(B3)	Associativity	$(a + b) + c = a + (b + c)$	$(a \cdot b) \cdot c = a \cdot (b \cdot c)$
(B4)	Absorption	$a + (a \cdot b) = a$	$a \cdot (a + b) = a$
(B5)	Distributivity	$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$	$a + (b \cdot c) = (a + b) \cdot (a + c)$
(B6)	Universal Bounds	$a + 0 = a,\ a + 1 = 1$	$a \cdot 1 = a,\ a \cdot 0 = 0$
(B7)	Complementary	$a + \overline{a} = 1$	$a \cdot \overline{a} = 0$
(B8)	Involution	$\overline{\overline{a}} = a$	
(B9)	Dualization	$\overline{a + b} = \overline{a} \cdot \overline{b}$	$\overline{a \cdot b} = \overline{a} + \overline{b}$

Properties (B1)-(B4) are common to every **lattice**,

i.e. a Boolean algebra is a distributive (B5), bounded (B6), and complemented (B7)-(B9) lattice,

i.e. every Boolean algebra can be characterized by a partial ordering on a set, *i.e.* $a \leq b$ if $a \cdot b = a$ or, alternatively, if $a + b = b$.

Set Theory, Boolean Algebra, Propositional Logic

Every theorem in one theory has a counterpart in each other theory.

Counterparts can be obtained applying the following substitutions:

Meaning	Set Theory	Boolean Algebra	Prop. Logic
values	2^X	B	$\mathcal{L}(V)$
"meet"/"and"	$\cap$	$\cdot$	$\wedge$
"join"/"or"	$\cup$	$+$	$\vee$
"complement"/"not"	^{c}	$^-$	$\neg$
identity element	X	1	1
zero element	$\emptyset$	0	0
partial order	$\subseteq$	$\leq$	$\rightarrow$

power set 2^X, set of logic variables V, set of all combinations $\mathcal{L}(V)$ of truth values of V

The Basic Principle of Classical Logic

The Principle of Bivalence:

"Every proposition is either true or false."

It has been formally developed by Tarski.

Alfred Tarski (1902-1983)

Łukasiewicz suggested to replace it by
The Principle of Valence:

"Every proposition has a truth value."

Propositions can have intermediate truth value,
expressed by a number from the unit interval $[0, 1]$.

Jan Łukasiewicz (1878-1956)

The Traditional or Aristotlelian Logic II

Short History

Aristotle introduced a logic of terms and drawing conclusion from two premises.

The great Greeks (Chrisippus) also developed logic of propositions.

Jan Łukasiewicz founded the multi-valued logic.

The multi-valued logic is to fuzzy set theory what classical logic is to set theory.

Three-valued Logics

A 2-valued logic can be extended to a 3-valued logic *in several ways*, *i.e.* different three-valued logics have been well established:

truth, falsity, indeterminacy are denoted by 1, 0, and 1/2, resp.

The negation $\neg a$ is defined as $1 - a$, *i.e.* $\neg 1 = 0, \neg 0 = 1$ and $\neg 1/2 = 1/2$.

Other primitives, *e.g.* $\wedge, \vee, \rightarrow, \leftrightarrow$, differ from logic to logic.

Five well-known three-valued logics (named after their originators) are defined in the following.

Primitives of Some Three-valued Logics

		Łukasiewicz				Bochvar				Kleene				Heyting				Reichenbach			
a	b	$\wedge$	$\vee$	$\rightarrow$	$\leftrightarrow$	$\wedge$	$\vee$	$\rightarrow$	$\leftrightarrow$	$\wedge$	$\vee$	$\rightarrow$	$\leftrightarrow$	$\wedge$	$\vee$	$\rightarrow$	$\leftrightarrow$	$\wedge$	$\vee$	$\rightarrow$	$\leftrightarrow$
0	0	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
0	$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	1	0	0	$\frac{1}{2}$	1	$\frac{1}{2}$
0	1	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0
$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$	1	1
$\frac{1}{2}$	1	$\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$
1	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0
1	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

All of them fully conform the usual definitions for $a, b \in \{0, 1\}$.

They differ from each other only in their treatment of $1/2$.

Question: Do they satisfy the law of contradiction $(a \wedge \neg a = 0)$ and the law of excluded middle $(a \vee \neg a = 1)$?

n-valued Logics

After the three-valued logics: generalizations to n-valued logics for arbitrary number of truth values $n \geq 2$.

In the 1930s, various n-valued logics were developed.

Usually truth values are assigned by rational number in $[0, 1]$.

Key idea: uniformly divide $[0, 1]$ into n truth values.

Definition
The set T_n of truth values of an n-valued logic is defined as

$$T_n = \left\{0 = \frac{0}{n-1}, \frac{1}{n-1}, \frac{2}{n-1}, \ldots, \frac{n-2}{n-1}, \frac{n-1}{n-1} = 1\right\}.$$

These values can be interpreted as degree of truth.

Primitives in n-valued Logics

Łukasiewicz proposed first series of n-valued logics for $n \geq 2$.

In the early 1930s, he simply generalized his three-valued logic.

It uses truth values in T_n and defines primitives as follows:

$$\neg a = 1 - a$$

$$a \wedge b = \min(a, b) \qquad a \to b = \min(1,\ 1 + b - a)$$

$$a \vee b = \max(a, b) \qquad a \leftrightarrow b = 1 - |a - b|$$

The n-valued logic of Łukasiewicz is denoted by L_n.

The sequence $(L_2, L_3, \ldots, L_\infty)$ contains the classical two-valued logic L_2 and an infinite-valued logic L_∞ (rational **countable** values T_∞).

The infinite-valued logic L_1 (**standard Łukasiewicz logic**) is the logic with all real numbers in $[0, 1]$ ($1 =$ cardinality of continuum).

From Logic to Fuzzy Logic

Zadeh's fuzzy logic proposal was much simpler

In 1965, he proposed a logic with values in $[0, 1]$:

$$\neg a = 1 - a,$$
$$a \wedge b = \min(a, b),$$
$$a \vee b = \max(a, b).$$

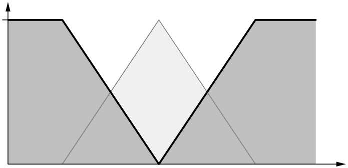
Zadeh in 2004

(born 1921)

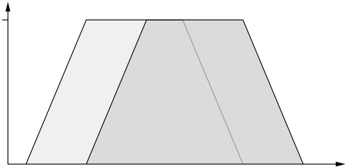
The set operators are defined pointwise as follows for μ, μ':

$$\neg\mu : X \to X, \neg\mu(x) = 1 - \mu(x),$$
$$\mu \wedge \mu' : X \to X (\mu \wedge \mu')(x) = \min\{\mu(x), \mu'(x)\},$$
$$\mu \vee \mu' : X \to X (\mu \vee \mu')(x) = \max\{\mu(x), \mu'(x)\}.$$

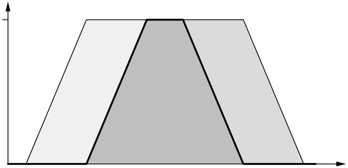
Standard Fuzzy Set Operators – Example



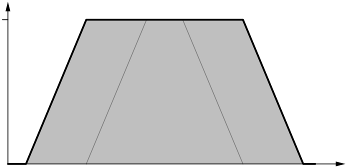
fuzzy complement



two fuzzy sets



fuzzy intersection



fuzzy union

Is Zadeh's logic a Boolean algebra?

Theorem

$(\mathcal{F}(X), \wedge, \vee, \neg)$ *is a complete distributive lattice but no Boolean algebra.*

Proof.

Consider $\mu : X \to X$ with $x \mapsto 0.5$, then $\neg\mu(x) = 0.5$ for all x and $\mu \wedge \neg\mu \neq \chi_{\emptyset}$. □

Fuzzy Set Theory

Definition
Let $X \neq \emptyset$ be a set.

$2^X \overset{\text{def}}{=} \{A \mid A \subseteq X\}$ power set of X,

$A \in 2^X, \quad \chi_A : X \to \{0, 1\}$ characteristic function,

$\mathcal{X}(X) \overset{\text{def}}{=} \left\{\chi_A \mid A \in 2^X\right\}$ set of characteristic functions.

Theorem

$(2^X, \cap, \cup, ^c)$ *is Boolean algebra,*

$\phi : 2^X \to \mathcal{X}(X), \quad \phi(A) \overset{def}{=} \chi_A$ *is bijection.*

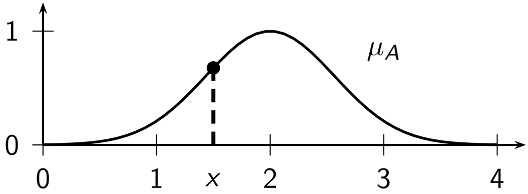
Theorem
$(\mathcal{X}(X), \wedge, \vee, \neg)$ *is Boolean algebra where*

$$\chi_{A \wedge B} \overset{def}{=} \min\{\chi_A, \chi_B\}, \quad \chi_{A \vee B} \overset{def}{=} \max\{\chi_A, \chi_B\}, \quad \chi_{\neg A} \overset{def}{=} 1 - \chi_A.$$

What does a fuzzy set represent?

Consider fuzzy proposition A ("approximately two") on $\mathbb{R}$

fuzzy logic offers means to construct such imprecise sentences



A defined by membership function μ_A, *i.e.* truth values $\forall x \in \mathbb{R}$

let $x \in \mathbb{R}$ be a subject/observation

$\mu_A(x)$ is the degree of truth that x is A

Standard Fuzzy Set Operators

Definition

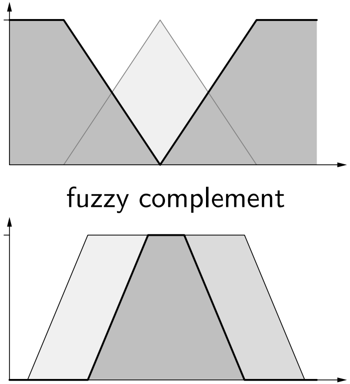
We define the following algebraicoperators on $\mathcal{F}(X)$:

$$(\mu \wedge \mu')(x) \stackrel{\text{def}}{=} \min\{\mu(x), \mu'(x)\} \qquad \text{intersection ("AND")},$$
$$(\mu \vee \mu')(x) \stackrel{\text{def}}{=} \max\{\mu(x), \mu'(x)\} \qquad \text{union ("OR")},$$
$$\neg\mu(x) \stackrel{\text{def}}{=} 1 - \mu(x) \qquad \text{complement ("NOT")}.$$

μ is subset of μ' if and only if $\mu \leq \mu'$.

Theorem
$(\mathcal{F}(X), \wedge, \vee, \neg)$ *is a complete distributive lattice but no boolean algebra.*

Standard Fuzzy Set Operators – Example



fuzzy complement

two fuzzy sets

fuzzy intersection

fuzzy union

Fuzzy Set Complement

Fuzzy Complement/Fuzzy Negation

Definition

Let X be a given set and $\mu \in \mathcal{F}(X)$. Then the *complement* $\bar{\mu}$ can be defined pointwise by $\bar{\mu}(x) := \sim(\mu(x))$ where $\sim\ : [0,1] \to [0,1]$ satisfies the conditions

$$\sim(0) = 1, \quad \sim(1) = 0$$

and

$$\text{for } x, y \in [0,1],\ x \le y \Longrightarrow\ \sim x \ge \sim y \quad (\sim \text{ is non-increasing}).$$

Abbreviation: $\sim x := \sim(x)$

Strict and Strong Negations

Additional properties may be required

- $x, y \in [0, 1],\ x < y \Longrightarrow\ \sim x > \sim y$ ($\sim$ is strictly decreasing)
- $\sim$ is continuous
- $\sim\sim x = x$ for all $x \in [0, 1]$ ($\sim$ is involutive)

According to conditions, two subclasses of negations are defined:

Definition
A negation is called *strict* if it is also strictly decreasing and continuous. A strict negation is said to be *strong* if it is involutive, too.

$\sim x = 1 - x^2$, for instance, is strict, not strong, thus not involutive

Families of Negations

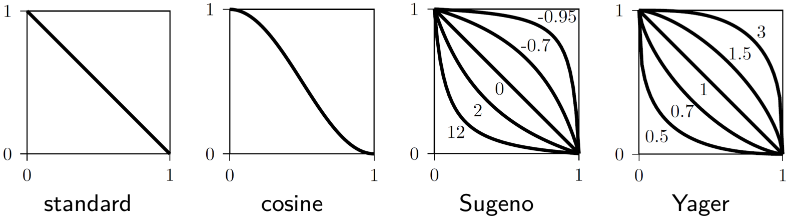
standard negation: $\sim x = 1 - x$

threshold negation: $\sim_\theta(x) = \begin{cases} 1 & \text{if } x \leq \theta \\ 0 & \text{otherwise} \end{cases}$

Cosine negation: $\sim x = \frac{1}{2}(1 + \cos(\pi x))$

Sugeno negation: $\sim_\lambda(x) = \frac{1 - x}{1 + \lambda x}, \quad \lambda > -1$

Yager negation: $\sim_\lambda(x) = (1 - x^\lambda)^{\frac{1}{\lambda}}$



Two Extreme Negations

$$\textit{intuitionistic}\text{ negation } \sim_i(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x > 0 \end{cases}$$

$$\textit{dual intuitionistic}\text{ negation } \sim_{di}(x) = \begin{cases} 1 & \text{if } x < 1 \\ 0 & \text{if } x = 1 \end{cases}$$

Both negations are not strictly increasing, not continuous, not involutive

Thus they are neither strict nor strong

They are "optimal" since their notions are nearest to crisp negation

$\sim_i$ and $\sim_{di}$ are two extreme cases of negations

For any negation $\sim$ the following holds

$$\sim_i \leq \sim \leq \sim_{di}$$

Inverse of a Strict Negation

Any strict negation $\sim$ is strictly decreasing and continuous.

Hence one can define its inverse $\sim^{-1}$.

$\sim^{-1}$ is also strict but in general differs from $\sim$.

$\sim^{-1} = \sim$ if and only if $\sim$ is involutive.

Every strict negation $\sim$ has a unique value $0 < s_\sim < 1$ such that $\sim s_\sim = s_\sim$.

$s_\sim$ is called *membership crossover point*.

$A(a) > s_\sim$ if and only if $A^c(a) < s_\sim$ where A^c is defined via $\sim$.

$\sim^{-1}(s_\sim) = s_\sim$ always holds as well.

Representation of Negations

Any strong negation can be obtained from standard negation.

Let $a, b \in \mathbb{R},\ a \leq b$.

Let $\varphi : [a, b] \to [a, b]$ be continuous and strictly increasing.

φ is called *automorphism* of the interval $[a, b] \subset \mathbb{R}$.

Theorem
A function $\sim\ : [0, 1] \to [0, 1]$ *is a strong negation if and only if there exists an automorphism* φ *of the unit interval such that for all* $x \in [0, 1]$ *the following holds*

$$\sim_\varphi(x) = \varphi^{-1}(1 - \varphi(x)).$$

$\sim_\varphi(x) = \varphi^{-1}(1 - \varphi(x))$ is called φ-transform of the standard negation.

Fuzzy Set Intersection and Union

Classical Intersection and Union

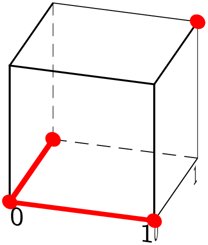
Classical set intersection represents logical conjunction.

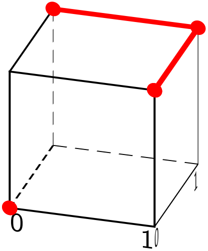
Classical set union represents logical disjunction.

Generalization from $\{0, 1\}$ to $[0, 1]$ as follows:

$x \wedge y$	0	1
0	0	0
1	0	1

$x \vee y$	0	1
0	0	1
1	1	1





Fuzzy Set Intersection and Union

Let A, B be fuzzy subsets of X, *i.e.* $A, B \in \mathcal{F}(X)$.

Their **intersection** and **union** can be defined pointwise using:

$$(A \cap B)(x) = \top(A(x), B(x)) \quad \text{where} \quad \top : [0,1]^2 \to [0,1]$$
$$(A \cup B)(x) = \bot(A(x), B(x)) \quad \text{where} \quad \bot : [0,1]^2 \to [0,1].$$

Triangular Norms and Conorms I

$\top$ is a *triangular norm (t-norm)* $\iff$ $\top$ satisfies conditions T1-T4

$\bot$ is a *triangular conorm (t-conorm)* $\iff$ $\bot$ satisfies C1-C4

for all $x, y \in [0, 1]$, the following laws hold

Identity Law
T1: $\top(x, 1) = x \quad (A \cap X = A)$
C1: $\bot(x, 0) = x \quad (A \cup \emptyset = A)$.

Commutativity
T2: $\top(x, y) = \top(y, x) \quad (A \cap B = B \cap A)$,
C2: $\bot(x, y) = \bot(y, x) \quad (A \cup B = B \cup A)$.

Triangular Norms and Conorms II

for all $x, y, z \in [0, 1]$, the following laws hold

Associativity
T3: $\top(x, \top(y, z)) = \top(\top(x, y), z) \quad (A \cap (B \cap C)) = ((A \cap B) \cap C)$,
C3: $\bot(x, \bot(y, z)) = \bot(\bot(x, y), z) \quad (A \cup (B \cup C)) = ((A \cup B) \cup C)$.

Monotonicity
$y \leq z$ implies
T4: $\top(x, y) \leq \top(x, z)$
C4: $\bot(x, y) \leq \bot(x, z)$.

Triangular Norms and Conorms III

$\top$ is a *triangular norm (t-norm)* $\Longleftrightarrow$ $\top$ satisfies conditions T1-T4

$\bot$ is a *triangular conorm (t-conorm)* $\Longleftrightarrow$ $\bot$ satisfies C1-C4

Both identity law and monotonicity respectively imply

$$\forall x \in [0,1] : \top(0,x) = 0,$$
$$\forall x \in [0,1] : \bot(1,x) = 1,$$
$$\text{for any } t\text{-norm } \top : \top(x,y) \leq \min(x,y),$$
$$\text{for any } t\text{-conorm } \bot : \bot(x,y) \geq \max(x,y).$$

note: $x = 1 \Rightarrow T(0,1) = 0$ and
$x \leq 1 \Rightarrow T(x,0) \leq T(1,0) = T(0,1) = 0$

De Morgan Triplet I

For every $\top$ and strong negation $\sim$, one can define *t*-conorm $\perp$ by

$$\perp(x, y) = \sim\top(\sim x, \sim y), \quad x, y \in [0, 1].$$

Additionally, in this case $\top(x, y) = \sim\perp(\sim x, \sim y),\ \ x, y \in [0, 1]$.

$\perp, \top$ are called *N-dual t-conorm* and *N-dual t-norm* to $\top, \perp$, resp.

In case of the standard negation $\sim x = 1 - x$ for $x \in [0, 1]$,
N-dual $\perp$ and $\top$ are called *dual t-conorm* and *dual t-norm*, resp.

$\perp(x, y) = \sim\top(\sim x, \sim y)$ expresses "fuzzy" De Morgan's law.

note: De Morgan's laws $(A \cup B)^c = A^c \cap B^c,\ (A \cap B)^c = A^c \cup B^c$

De Morgan Triplet II

Definition
The triplet $(\top, \bot, \sim)$ is called *De Morgan triplet* if and only if

$\top$ is t-norm, $\bot$ is t-conorm, $\sim$ is strong negation,

$\top, \bot$ and $\sim$ satisfy $\bot(x, y) = \sim \top(\sim x, \sim y)$.

In the following, some important De Morgan triplets will be shown,

only the most frequently used and important ones.

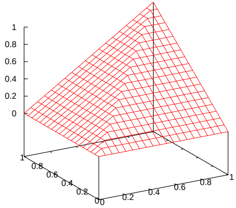
In all cases, the standard negation $\sim x = 1 - x$ is considered.

The Minimum and Maximum I

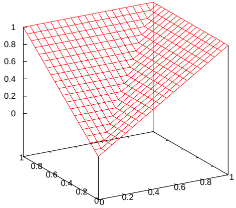
$\top_{\min}(x, y) = \min(x, y), \quad \bot_{\max}(x, y) = \max(x, y)$

Minimum is the greatest t-norm and max is the weakest t-conorm.

$\top(x, y) \leq \min(x, y)$ and $\bot(x, y) \geq \max(x, y)$ for any $\top$ and $\bot$



$\top_{\min}$



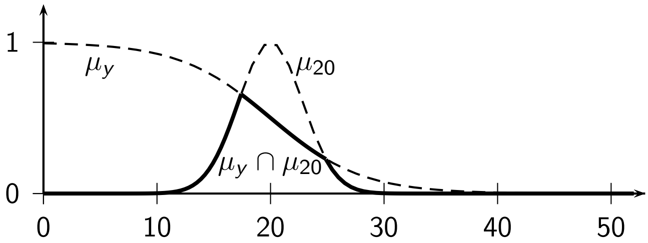
$\bot_{\max}$

The Minimum and Maximum II

$\top_{\min}$ and $\bot_{\max}$ can be easily processed numerically and visually,

e.g. linguistic values *young* and *approx. 20* described by μ_y, μ_{20}.

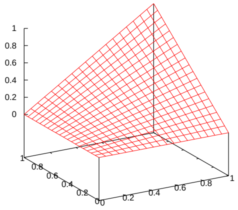
$\top_{\min}(\mu_y, \mu_{20})$ is shown below.



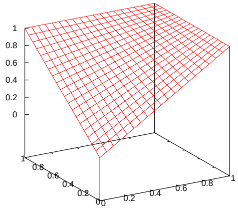
The Product and Probabilistic Sum

$\top_{\text{prod}}(x, y) = x \cdot y, \quad \bot_{\text{sum}}(x, y) = x + y - x \cdot y$

Note that use of product and its dual has nothing to do with probability theory.



$\top_{\text{prod}}$

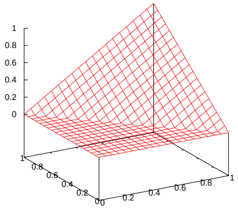


$\bot_{\text{sum}}$

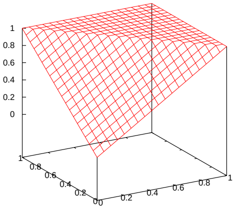
The Łukasiewicz t-norm and t-conorm

$$\top_{\text{Łuka}}(x, y) = \max\{0,\ x + y - 1\}, \quad \bot_{\text{Łuka}}(x, y) = \min\{1,\ x + y\}$$

$\top_{\text{Łuka}}$, $\bot_{\text{Łuka}}$ are also called *bold intersection* and *bounded sum*.



$\top_{\text{Łuka}}$



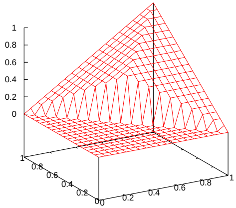
$\bot_{\text{Łuka}}$

The Nilpotent Minimum and Maximum

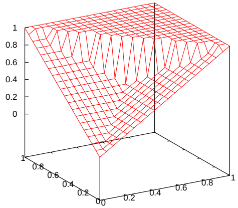
$$\top_{\min_0}(x,y) = \begin{cases} \min(x,y) & \text{if } x+y>1 \\ 0 & \text{otherwise} \end{cases}$$

$$\bot_{\max_1}(x,y) = \begin{cases} \max(x,y) & \text{if } x+y<1 \\ 1 & \text{otherwise} \end{cases}$$

Found in 1992 and independently rediscovered in 1995.



$\top_{\min_0}$



$\bot_{\max_1}$

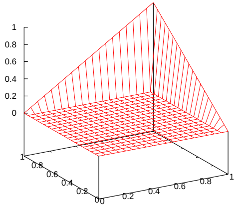
The Drastic Product and Sum

$$\top_{-1}(x, y) = \begin{cases} \min(x, y) & \text{if } \max(x, y) = 1 \\ 0 & \text{otherwise} \end{cases}$$

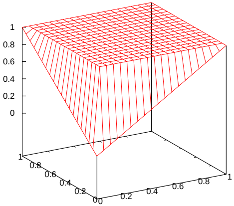
$$\bot_{-1}(x, y) = \begin{cases} \max(x, y) & \text{if } \min(x, y) = 0 \\ 1 & \text{otherwise} \end{cases}$$

$\top_{-1}$ is the weakest *t*-norm, $\bot_{-1}$ is the strongest *t*-conorm.

$\top_{-1} \leq \top \leq \top_{\min}, \quad \bot_{\max} \leq \bot \leq \bot_{-1}$ for any $\top$ and $\bot$

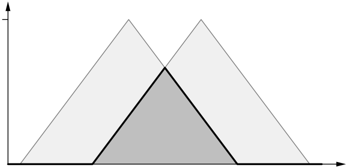


$\top_{-1}$

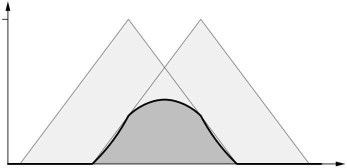


$\bot_{-1}$

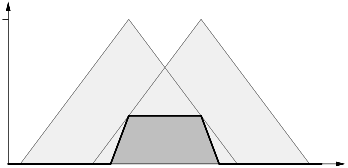
Examples of Fuzzy Intersections



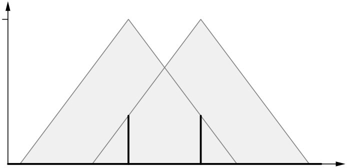
t-norm $\top_{\min}$



t-norm $\top_{\text{prod}}$



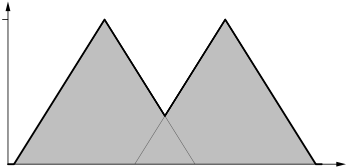
t-norm $\top_{\text{Łuka}}$



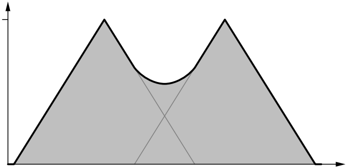
t-norm $\top_{-1}$

Note that all fuzzy intersections are contained within upper left graph and lower right one.

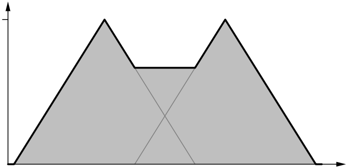
Examples of Fuzzy Unions



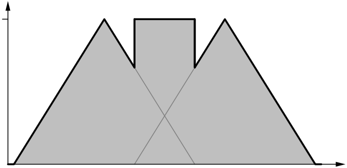
t-conorm $\bot_{\max}$



t-conorm $\bot_{\text{sum}}$



t-conorm $\bot_{\text{Łuka}}$



t-conorm $\bot_{-1}$

Note that all fuzzy unions are contained within upper left graph and lower right one.

The Special Role of Minimum and Maximum I

$\top_{\min}$ and $\bot_{\max}$ play key role for intersection and union, resp.

In a practical sense, they are very simple.

Apart from the identity law, commutativity, associativity and monotonicity, they also satisfy the following properties for all $x, y, z \in [0, 1]$:

Distributivity

$$\bot_{\max}(x, \top_{\min}(y, z)) = \top_{\min}(\bot_{\max}(x, y), \bot_{\max}(x, z)),$$
$$\top_{\min}(x, \bot_{\max}(y, z)) = \bot_{\max}(\top_{\min}(x, y), \top_{\min}(x, z))$$

Continuity

$\top_{\min}$ and $\bot_{\max}$ are continuous.

The Special Role of Minimum and Maximum II

Strict monotonicity on the diagonal

$x < y$ implies $\top_{\min}(x, x) < \top_{\min}(y, y)$ and $\bot_{\max}(x, x) < \bot_{\max}(y, y)$.

Idempotency

$\top_{\min}(x, x) = x, \quad \bot_{\max}(x, x) = x$

Absorption

$\top_{\min}(x, \bot_{\max}(x, y)) = x, \quad \bot_{\max}(x, \top_{\min}(x, y)) = x$

Non-compensation

$x < y < z$ imply $\top_{\min}(x, z) \neq \top_{\min}(y, y)$ and $\bot_{\max}(x, z) \neq \bot_{\max}(y, y)$.

The Special Role of Minimum and Maximum III

Is $(\mathcal{F}(X), \top_{\text{min}}, \bot_{\text{max}}, \sim)$ **a boolean algebra?**

Consider the properties (B1)-(B9) of any Boolean algebra.

For $(\mathcal{F}(X), \top_{\text{min}}, \bot_{\text{max}}, \sim)$ with strong negation $\sim$ only complementary (B7) does not hold.

Hence $(\mathcal{F}(X), \top_{\text{min}}, \bot_{\text{max}}, \sim)$ is a *completely distributive lattice* with identity element μ_X and zero element $\mu_\emptyset$.

No lattice $(\mathcal{F}(X), \top, \bot, \sim)$ forms a Boolean algebra

due to the fact that complementary (B7) does not hold:

- There is no complement/negation $\sim$ with $\top(A, \sim A) = \mu_\emptyset$.
- There is no complement/negation $\sim$ with $\bot(A, \sim A) = \mu_X$.

Complementary Property of Fuzzy Sets

Using fuzzy sets, it's **impossible** to keep up a Boolean algebra.

Verify, *e.g.* that law of contradiction is violated, *i.e.*

$$(\exists x \in X)(A \cap A^c)(x) \neq \emptyset.$$

We use $\min, \max$ and strong negation $\sim$ as fuzzy set operators.

So we need to show that

$$\min\{A(x), 1 - A(x)\} = 0$$

is violated for at least one $x \in X$.

easy: This Equation is violated for all $A(x) \in (0, 1)$.

It is satisfied only for $A(x) \in \{0, 1\}$.

The concept of a pseudoinverse

Definition

Let $f : [a, b] \to [c, d]$ be a monotone function between two closed subintervals of extended real line. The pseudoinverse function to f is the function $f^{(-1)} : [c, d] \to [a, b]$ defined as

$$f^{(-1)}(y) = \begin{cases} \sup\{x \in [a, b] \mid f(x) < y\} & \text{for } f \text{ non-decreasing,} \\ \sup\{x \in [a, b] \mid f(x) > y\} & \text{for } f \text{ non-increasing.} \end{cases}$$

Continuous Archimedean t-norms and t-conorms

broad class of problems relates to representation of multi-place functions by composition of a "simpler" function, *e.g.*

$$K(x, y) = f^{(-1)}(f(x) + f(y))$$

So, one should consider suitable subclass of all t-norms.

Definition
A t-norm $\top$ is
(a) *continuous* if $\top$ as function is continuous on unit interval,

(b) *Archimedean* if $\top$ is continuous and $\top(x, x) < x$ for all $x \in]0, 1[$.

Definition
A t-conorm $\perp$ is
(a) *continuous* if $\perp$ as function is continuous on unit interval,

(b) *Archimedean* if $\perp$ is continuous and $\perp(x, x) > x$ for all $x \in]0, 1[$.

Continuous Archimedean t-norms

Theorem
A t-norm $\top$ is continuous and Archimedean if and only if there exists a strictly decreasing and continuous function $f : [0,1] \to [0,\infty]$ with $f(1) = 0$ such that

$$\top(x,y) = f^{(-1)}(f(x) + f(y)) \tag{1}$$

where

$$f^{(-1)}(x) = \begin{cases} f^{-1}(x) & \text{if } x \leq f(0) \\ 0 & \text{otherwise} \end{cases}$$

is the pseudoinverse of f. Moreover, this representation is unique up to a positive multiplicative constant.

$\top$ *is generated by* f if $\top$ has representation (1).

f is called *additive generator* of $\top$.

Additive Generators of t-norms – Examples

Find an additive generator f of $\top_{\text{Łuka}}(x, y) = \max\{x + y - 1,\ 0\}$.

for instance $f_{\text{Łuka}}(x) = 1 - x$

then, $f^{(-1)}_{\text{Łuka}}(x) = \max\{1 - x,\ 0\}$

thus $\top_{\text{Łuka}}(x, y) = f^{(-1)}_{\text{Łuka}}(f_{\text{Łuka}}(x) + f_{\text{Łuka}}(y))$

Find an additive generator f of $\top_{\text{prod}}(x, y) = x \cdot y$.

to be discussed in the exercise

hint: use of logarithmic and exponential function

Continuous Archimedean t-conorms

Theorem
A t-conorm $\bot$ is continuous and Archimedean if and only if there exists a strictly increasing and continuous function $g : [0,1] \to [0,\infty]$ with $g(0) = 0$ such that

$$\bot(x, y) = g^{(-1)}(g(x) + g(y)) \tag{2}$$

where

$$g^{(-1)}(x) = \begin{cases} g^{-1}(x) & \text{if } x \le g(1) \\ 1 & \text{otherwise} \end{cases}$$

is the pseudoinverse of g. Moreover, this representation is unique up to a positive multiplicative constant.

$\bot$ *is generated by g* if $\bot$ has representation (2).

g is called *additive generator* of $\bot$.

Additive Generators of t-conorms – Two Examples

Find an additive generator g of $\bot_{\text{Łuka}}(x, y) = \min\{x + y,\ 1\}$.

for instance $g_{\text{Łuka}}(x) = x$

then, $g^{(-1)}_{\text{Łuka}}(x) = \min\{x,\ 1\}$

thus $\bot_{\text{Łuka}}(x, y) = g^{(-1)}_{\text{Łuka}}(g_{\text{Łuka}}(x) + g_{\text{Łuka}}(y))$

Find an additive generator g of $\bot_{\text{sum}}(x, y) = x + y - x \cdot y$.

to be discussed in the exercise

hint: use of logarithmic and exponential function

Now, let us examine some typical families of operations.

Hamacher Family I

$$\top_\alpha(x,y) = \frac{x \cdot y}{\alpha + (1-\alpha)(x + y + x \cdot y)}, \quad \alpha \geq 0,$$
$$\bot_\beta(x,y) = \frac{x + y + (\beta - 1) \cdot x \cdot y}{1 + \beta \cdot x \cdot y}, \quad \beta \geq -1,$$
$$\sim_\gamma(x) = \frac{1-x}{1+\gamma x}, \quad \gamma > -1$$

Theorem

$(\top, \bot, \sim)$ is a De Morgan triplet such that

$$\top(x,y) = \top(x,z) \Longrightarrow y = z,$$
$$\bot(x,y) = \bot(x,z) \Longrightarrow y = z,$$
$$\forall z \leq x \ \exists y, y' \text{ such that } \top(x,y) = z, \ \bot(z,y') = x$$

and $\top$ and $\bot$ are rational functions if and only if there are numbers $\alpha \geq 0, \ \beta \geq -1$ and $\gamma > -1$ such that $\alpha = \frac{1+\beta}{1+\gamma}$ and $\top = \top_\alpha, \ \bot = \bot_\beta$ and $\sim = \sim_\gamma$.

Hamacher Family II

Additive generators f_α of $\top_\alpha$ are

$$f_\alpha = \begin{cases} \frac{1-x}{x} & \text{if } \alpha = 0 \\ \log \frac{\alpha+(1-\alpha)x}{x} & \text{if } \alpha > 0. \end{cases}$$

Each member of these families is strict t-norm and strict t-conorm, respectively.

Members of this family of t-norms are decreasing functions of parameter α.

Sugeno-Weber Family I

For $\lambda > 1$ and $x, y \in [0, 1]$, define

$$\top_\lambda(x, y) = \max\left\{\frac{x + y - 1 + \lambda xy}{1 + \lambda},\ 0\right\},$$
$$\bot_\lambda(x, y) = \min\{x + y + \lambda xy,\ 1\}.$$

$\lambda = 0$ leads to $\top_{\text{Łuka}}$ and $\bot_{\text{Łuka}}$, resp.

$\lambda \to \infty$ results in $\top_{\text{prod}}$ and $\bot_{\text{sum}}$, resp.

$\lambda \to -1$ creates $\top_{-1}$ and $\bot_{-1}$, resp.

Sugeno-Weber Family II

Additive generators f_λ of $\top_\lambda$ are

$$f_\lambda(x) = \begin{cases} 1 - x & \text{if } \lambda = 0 \\ 1 - \frac{\log(1+\lambda x)}{\log(1+\lambda)} & \text{otherwise.} \end{cases}$$

$\{\top_\lambda\}_{\lambda > -1}$ are increasing functions of parameter λ.

Additive generators of $\bot_\lambda$ are $g_\lambda(x) = 1 - f_\lambda(x)$.

Yager Family

For $0 < p < \infty$ and $x, y \in [0, 1]$, define

$$\top_p(x, y) = \max\left\{1 - ((1-x)^p + (1-y)^p)^{1/p},\ 0\right\},$$
$$\bot_p(x, y) = \min\left\{(x^p + y^p)^{1/p},\ 1\right\}.$$

Additive generators of $\top_p$ are

$$f_p(x) = (1-x)^p,$$

and of $\bot_p$ are

$$g_p(x) = x^p.$$

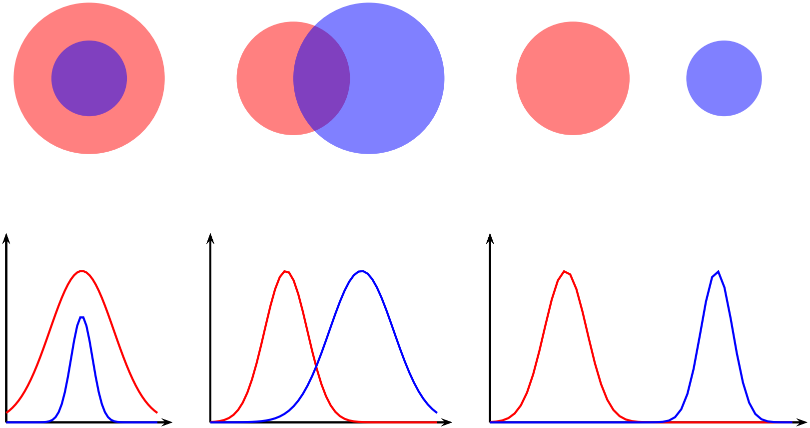
$\{\top_p\}_{0<p<\infty}$ are strictly increasing in p.

Note that $\lim_{p\to+0} \top_p = \top_{\text{Łuka}}$.

Fuzzy Sets Inclusion

Fuzzy Implications

crisp: $x \in A \Rightarrow x \in B$, **fuzzy:** $x \in \mu \Rightarrow x \in \mu'$



Definitions of Fuzzy Implications

One way of defining I is to use $\forall a, b \in \{0, 1\}$

$$I(a, b) = \neg a \vee b.$$

In fuzzy logic, disjunction and negation are t-conorm and fuzzy complement, resp., thus $\forall a, b \in [0, 1]$

$$I(a, b) = \bot(\sim a, b).$$

Another way in classical logic is $\forall a, b \in \{0, 1\}$

$$I(a, b) = \max\left\{x \in \{0, 1\} \mid a \wedge x \leq b\right\}.$$

In fuzzy logic, conjunction represents t-norm, thus $\forall a, b \in [0, 1]$

$$I(a, b) = \sup\left\{x \in [0, 1] \mid \top(a, x) \leq b\right\}.$$

So, classical definitions are equal, fuzzy extensions are not.

Definitions of Fuzzy Implications

$I(a, b) = \bot(\sim a, b)$ may also be written as either

$$I(a, b) = \neg a \vee (a \wedge b) \qquad \text{or}$$
$$I(a, b) = (\neg a \wedge \neg b) \vee b.$$

Fuzzy logical extensions are thus, respectively,

$$I(a, b) = \bot(\sim a, \top(a, b)),$$
$$I(a, b) = \bot(\top(\sim a, \sim b), b)$$

where $(\top, \bot, n)$ must be a *De Morgan triplet*.

So again, classical definitions are equal, fuzzy extensions are not.

reason: Law of absorption of negation does not hold in fuzzy logic.

S-Implications

Implications based on $I(a, b) = \bot(\sim a, b)$ are called S-**implications**.

Symbol S is often used to denote t-conorms.

Four well-known S-implications are based on $\sim a = 1 - a$:

Name	$I(a, b)$	$\bot(a, b)$
Kleene-Dienes	$I_{\max}(a, b) = \max(1 - a, b)$	$\max(a, b)$
Reichenbach	$I_{\text{sum}}(a, b) = 1 - a + ab$	$a + b - ab$
Łukasiewicz	$I_{Ł}(a, b) = \min(1,\ 1 - a + b)$	$\min(1,\ a + b)$
largest	$I_{-1}(a, b) = \begin{cases} b, & \text{if } a = 1 \\ 1 - a, & \text{if } b = 0 \\ 1, & \text{otherwise} \end{cases}$	$\begin{cases} b, & \text{if } a = 0 \\ a, & \text{if } b = 0 \\ 1, & \text{otherwise} \end{cases}$

S-Implications

The drastic sum $\bot_{-1}$ leads to the largest S-implication I_{-1} due to the following theorem:

Theorem
Let $\bot_1, \bot_2$ be t-conorms such that $\bot_1(a,b) \leq \bot_2(a,b)$ for all $a, b \in [0,1]$. Let I_1, I_2 be S-implications based on same fuzzy complement $\sim$ and $\bot_1, \bot_2$, respectively. Then $I_1(a,b) \leq I_2(a,b)$ for all $a, b \in [0,1]$.

Since $\bot_{-1}$ leads to the largest S-implication, similarly, $\bot_{\max}$ leads to the smallest S-implication $I_{\max}$.

Furthermore,

$$I_{\max} \leq I_{\text{sum}} \leq I_{Ł} \leq I_{-1}.$$

R-Implications

$I(a, b) = \sup\{x \in [0,1] \mid \top(a, x) \leq b\}$ leads to R-**implications**.

Symbol R represents close connection to residuated semigroup.

Three well-known R-implications are based on $\sim a = 1 - a$:

- Standard fuzzy intersection leads to **Gödel implication**

$$I_{\min}(a, b) = \sup\{x \mid \min(a, x) \leq b\} = \begin{cases} 1, & \text{if } a \leq b \\ b, & \text{if } a > b. \end{cases}$$

- Product leads to **Goguen implication**

$$I_{\text{prod}}(a, b) = \sup\{x \mid ax \leq b\} = \begin{cases} 1, & \text{if } a \leq b \\ b/a, & \text{if } a > b. \end{cases}$$

- Łukasiewicz t-norm leads to **Łukasiewicz implication**

$$I_{Ł}(a, b) = \sup\{x \mid \max(0, a + x - 1) \leq b\} = \min(1,\ 1 - a + b).$$

R-Implications

Name	Formula	$\top(a, b) =$
Gödel	$I_{\min}(a, b) = \begin{cases} 1, & \text{if } a \leq b \\ b, & \text{if } a > b \end{cases}$	$\min(a, b)$
Goguen	$I_{\text{prod}}(a, b) = \begin{cases} 1, & \text{if } a \leq b \\ b/a, & \text{if } a > b \end{cases}$	ab
Łukasiewicz	$I_{Ł}(a, b) = \min(1,\ 1 - a + b)$	$\max(0,\ a + b - 1)$
largest	$I_{\text{L}}(a, b) = \begin{cases} b, & \text{if } a = 1 \\ 1, & \text{otherwise} \end{cases}$	not defined

I_{L} is actually the limit of all R-implications.

It serves as least upper bound.

It cannot be defined by $I(a, b) = \sup\{x \in [0, 1] \mid \top(a, x) \leq b\}$.

R-Implications

Theorem
Let $\top_1, \top_2$ be t-norms such that $\top_1(a, b) \leq \top_2(a, b)$ for all $a, b \in [0, 1]$. Let I_1, I_2 be R-implications based on $\top_1, \top_2$, respectively. Then $I_1(a, b) \geq I_2(a, b)$ for all $a, b \in [0, 1]$.

It follows that Gödel $I_{\min}$ is the smallest R-implication.

Furthermore,

$$I_{\min} \leq I_{\text{prod}} \leq I_{Ł} \leq I_{L}.$$

QL-Implications

Implications based on $I(a, b) = \bot(\sim a, \top(a, b))$ are called QL-**implications** (QL from quantum logic).

Four well-known QL-implications are based on $\sim a = 1 - a$:

- Standard min and max lead to **Zadeh implication**

$$I_Z(a, b) = \max[1 - a, \min(a, b)].$$

- The algebraic product and sum lead to

$$I_{\mathrm{p}}(a, b) = 1 - a + a^2 b.$$

- Using $\top_{Ł}$ and $\bot_{Ł}$ leads to **Kleene-Dienes implication** again.
- Using $\top_{-1}$ and $\bot_{-1}$ leads to

$$I_{\mathrm{q}}(a, b) = \begin{cases} b, & \text{if } a = 1 \\ 1 - a, & \text{if } a \neq 1, b \neq 1 \\ 1, & \text{if } a \neq 1, b = 1. \end{cases}$$

Axioms

All I come from generalizations of the classical implication.

They collapse to the classical implication when truth values are 0 or 1.

Generalizing classical properties leads to following axioms:

1) $a \leq b$ implies $I(a, x) \geq I(b, x)$ *(monotonicity in 1st argument)*
2) $a \leq b$ implies $I(x, a) \leq I(x, b)$ *(monotonicity in 2nd argument)*
3) $I(0, a) = 1$ *(dominance of falsity)*
4) $I(1, b) = b$ *(neutrality of truth)*
5) $I(a, a) = 1$ *(identity)*
6) $I(a, I(b, c)) = I(b, I(a, c))$ *(exchange property)*
7) $I(a, b) = 1$ if and only if $a \leq b$ *(boundary condition)*
8) $I(a, b) = I(\sim b, \sim a)$ for fuzzy complement $\sim$ *(contraposition)*
9) I is a continuous function *(continuity)*

Generator Function

I that satisfy all listed axioms are characterized by this theorem:

Theorem
A function $I : [0,1]^2 \to [0,1]$ *satisfies Axioms 1–9 of fuzzy implications for a particular fuzzy complement* $\sim$ *if and only if there exists a strict increasing continuous function* $f : [0,1] \to [0,\infty)$ *such that* $f(0) = 0$,

$$I(a,b) = f^{(-1)}(f(1) - f(a) + f(b))$$

for all $a, b \in [0,1]$, *and*

$$\sim a = f^{-1}(f(1) - f(a))$$

for all $a \in [0,1]$.

Example

Consider $f_\lambda(a) = \ln(1 + \lambda a)$ with $a \in [0, 1]$ and $\lambda > 0$.

Its pseudo-inverse is

$$f_\lambda^{(-1)}(a) = \begin{cases} \frac{e^a - 1}{\lambda}, & \text{if } 0 \leq a \leq \ln(1 + \lambda) \\ 1, & \text{otherwise.} \end{cases}$$

The fuzzy complement generated by f for all $a \in [0, 1]$ is

$$n_\lambda(a) = \frac{1 - a}{1 + \lambda a}.$$

The resulting fuzzy implication for all $a, b \in [0, 1]$ is thus

$$I_\lambda(a, b) = \min\left(1, \ \frac{1 - a + b + \lambda b}{1 + \lambda a}\right).$$

If $\lambda \in (-1, 0)$, then I_λ is called **pseudo-Łukasiewicz implication**.

List of Implications in Many Valued Logics

Name	Class	Form $I(a, b) =$	Axioms	Complement
Gaines-Rescher		$\begin{cases} 1 & \text{if } a \leq b \\ 0 & \text{otherwise} \end{cases}$	1–8	$1 - a$
Gödel	R	$\begin{cases} 1 & \text{if } a \leq b \\ b & \text{otherwise} \end{cases}$	1–7	
Goguen	R	$\begin{cases} 1 & \text{if } a \leq b \\ b/a & \text{otherwise} \end{cases}$	1–7, 9	
Kleene-Dienes	S,QL	$\max(1 - a, b)$	1–4, 6, 8, 9	$1 - a$
Łukasiewicz	R, S	$\min(1, 1 - a + b)$	1–9	$1 - a$
Pseudo-Łukasiewicz 1	R, S	$\min\left[1, \frac{1-a+(1+\lambda)b}{1+\lambda a}\right]$	1–9	$\frac{1-a}{1+\lambda a}$, $(\lambda > -1)$
Pseudo-Łukasiewicz 2	R, S	$\min\left[1, 1 - a^w + b^w\right]$	1–9	$(1 - a^w)^{\frac{1}{w}}$, $(w > 0)$
Reichenbach	S	$1 - a + ab$	1–4, 6, 8, 9	$1 - a$
Wu		$\begin{cases} 1 & \text{if } a \leq b \\ \min(1 - a, b) & \text{otherwise} \end{cases}$	1–3,5,7,8	$1 - a$
Zadeh	QL	$\max[1 - a, \min(a, b)]$	1–4, 9	$1 - a$

Which Fuzzy Implication?

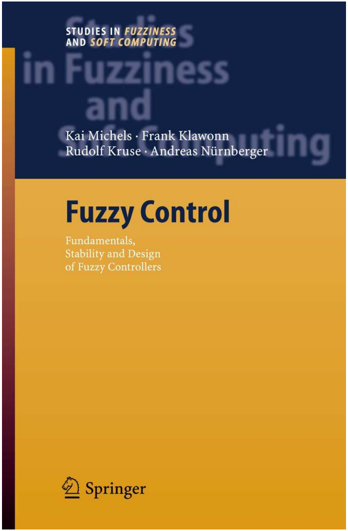
Since the meaning of I is not unique, we must resolve the following question:

Which I should be used for calculating the fuzzy relation R?

Hence meaningful criteria are needed.

They emerge from various fuzzy inference rules, *i.e.* modus ponens, modus tollens, hypothetical syllogism.

References I



References II

Hájek, P. and METAMATHEMATICS, O. F. L. (1998).
Trends in logic.
Metamathematics of Fuzzy Logic. Kluwer Academic, Dordrecht.

Klement, E. P., Mesiar, R., and Pap, E. (2004).
Triangular norms. position paper i: basic analytical and algebraic properties.
Fuzzy Sets and Systems, 143(1):5–26.

Klir, G. and Yuan, B. (1995).
Fuzzy sets and fuzzy logic, volume 4.
Prentice hall New Jersey.

Kruse, R., Borgelt, C., Braune, C., Mostaghim, S., and Steinbrecher, M. (2016).
Computational intelligence: a methodological introduction.
Springer.

Kruse, R., Gebhardt, J., and Klawonn, F. (1994).
Foundations of Fuzzy Systems.
John Wiley & Sons Ltd, Chichester, United Kingdom.

References III

Michels, K., Klawonn, F., Kruse, R., and Nürnberger, A. (2006).
Fuzzy Control: Fundamentals, Stability and Design of Fuzzy Controllers, volume 200 of *Studies in Fuzziness and Soft Computing*.
Springer, Berlin / Heidelberg, Germany.

Zadeh, L. A. (1965).
Fuzzy sets.
Information and control, 8(3):338–353.