OTTO VON GUERICKE UNIVERSITÄT MAGDEBURG

INF FAKULTÄT FÜR INFORMATIK

Fuzzy Systems

Approximate Reasoning

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The Extension Principle

Motivation I

How to extend $\phi : X^n \to Y$ to $\hat{\phi} : \mathcal{F}(X)^n \to \mathcal{F}(Y)$?

Let $\mu \in \mathcal{F}(\mathrm{I\!R})$ be a fuzzy set of the imprecise concept "about 2".

Then the degree of membership $\mu(2.2)$ can be seen as *truth value* of the statement "2.2 is about equal to 2".

Let $\mu' \in \mathcal{F}(\mathrm{I\!R})$ be a fuzzy set of the imprecise concept "old".

Then the truth value of "2.2 is about equal 2 **and** 2.2 is old" can be seen as membership degree of 2.2 *w.r.t.* imprecise concept "about 2 and old".

Motivation II – Operating on Truth Values

Any $\top$ ($\bot$) can be used to represent conjunction (disjunction).

However, now only $\top_{\min}$ and $\bot_{\max}$ shall be used.

Let $\mathcal{P}$ be set of imprecise statements that can be combined by *and*, *or*.

truth : $\mathcal{P} \to [0, 1]$ assigns truth value truth(a) to every $a \in \mathcal{P}$.

truth(a) $= 0$ means a is definitely false.

truth(a) $= 1$ means a is definitely true.

If $0 <$ truth(a) < 1, then only gradual truth of statement a.

Motivation III – Extension Principle

Combination of two statements $a, b \in P$:

$$\begin{aligned} \text{truth}(a \text{ and } b) = \quad & \text{truth}(a \wedge b) = \quad \min\{\text{truth}(a), \text{truth}(b)\}, \\ \text{truth}(a \text{ or } b) = \quad & \text{truth}(a \vee b) = \quad \max\{\text{truth}(a), \text{truth}(b)\} \end{aligned}$$

For infinite number of statements $a_i, i \in I$:

$$\begin{aligned} \text{truth}(\forall i \in I : a_i) &= \inf \{\text{truth}(a_i) \mid i \in I\}, \\ \text{truth}(\exists i \in I : a_i) &= \sup \{\text{truth}(a_i) \mid i \in I\} \end{aligned}$$

This concept helps to extend $\phi : X^n \to Y$ to $\hat{\phi} : \mathcal{F}(X)^n \to \mathcal{F}(Y)$.

- Crisp tuple $(x_1, \ldots, x_n)$ is mapped to crisp value $\phi(x_1, \ldots, x_n)$.
- Imprecise descriptions $(\mu_1, \ldots, \mu_n)$ of $(x_1, \ldots, x_n)$ are mapped to fuzzy value $\hat{\phi}(\mu_1, \ldots, \mu_n)$.

Example – How to extend the addition?

$+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \quad (a, b) \mapsto a + b$

Extensions to sets: $+ : 2^{\mathbb{R}} \times 2^{\mathbb{R}} \to 2^{\mathbb{R}}$

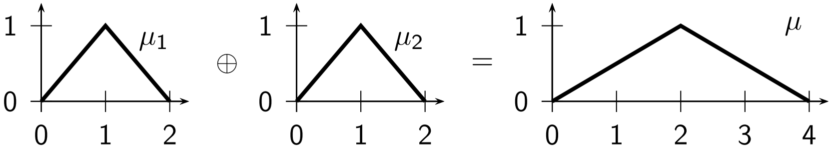
$$(A, B) \mapsto A + B = \{y \mid (\exists a)(\exists b)(y = a + b) \land (a \in A) \land (b \in B)\}$$

Extensions to fuzzy sets:

$$+ : \mathcal{F}(\mathbb{R}) \times \mathcal{F}(\mathbb{R}) \to \mathcal{F}(\mathbb{R}), \quad (\mu_1, \mu_2) \mapsto \mu_1 \oplus \mu_2$$

$$\begin{aligned}
\text{truth}(y \in \mu_1 \oplus \mu_2) &= \text{truth}((\exists a)(\exists b) : (y = a + b) \land (a \in \mu_1) \land (b \in \mu_2)) \\
&= \sup_{a,b}\{\text{truth}(y = a + b) \land \text{truth}(a \in \mu_1) \land \\
&\qquad \text{truth}(b \in \mu_2)\} \\
&= \sup_{a,b: y = a+b} \{\min(\mu_1(a),\ \mu_2(b))\}
\end{aligned}$$

Example – How to extend the addition?



$\mu(2) = 1$ because $\mu_1(1) = 1$ and $\mu_2(1) = 1$

$\mu(5) = 0$ because if $a + b = 5$, then $\min\{\mu_1(a),\ \mu_2(b)\} = 0$

$\mu(1) = 0.5$ because it is the result of an optimization task with optimum at, *e.g.* $a = 0.5$ and $b = 0.5$

Extension to Sets

Definition

Let $\phi : X^n \to Y$ be a mapping. The *extension* $\hat{\phi}$ of ϕ is given by

$$\hat{\phi} : [2^X]^n \to 2^Y \quad \text{with}$$

$$\hat{\phi}(A_1, \ldots, A_n) = \{y \in Y \mid \exists (x_1, \ldots, x_n) \in A_1 \times \ldots \times A_n : \phi(x_1, \ldots, x_n) = y\}.$$

Extension to Fuzzy Sets

Definition

Let $\phi : X^n \to Y$ be a mapping. The *extension* $\hat{\phi}$ of ϕ is given by

$$\hat{\phi} : [\mathcal{F}(X)]^n \to \mathcal{F}(Y) \quad \text{with}$$

$$\hat{\phi}(\mu_1, \ldots, \mu_n)(y) = \sup\{\min\{\mu_1(x_1), \ldots, \mu_n(x_n)\} \mid (x_1, \ldots, x_n) \in X^n \wedge \phi(x_1, \ldots, x_n) = y\}$$

assuming that $\sup \emptyset = 0$.

Example I

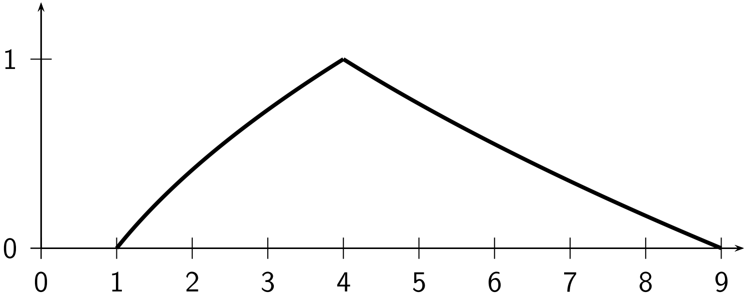
Let fuzzy set "approximately 2" be defined as

$$\mu(x) = \begin{cases} x - 1, & \text{if } 1 \leq x \leq 2 \\ 3 - x, & \text{if } 2 \leq x \leq 3 \\ 0, & \text{otherwise.} \end{cases}$$

The extension of $\phi : \mathbb{R} \to \mathbb{R}, x \mapsto x^2$ to fuzzy sets on $\mathbb{R}$ is

$$\begin{aligned} \hat{\phi}(\mu)(y) &= \sup \left\{ \mu(x) \;\middle|\; x \in \mathbb{R} \wedge x^2 = y \right\} \\ &= \begin{cases} \sqrt{y} - 1, & \text{if } 1 \leq y \leq 4 \\ 3 - \sqrt{y}, & \text{if } 4 \leq y \leq 9 \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Example II



The extension principle is taken as basis for “fuzzifying” whole theories.

Now, it will be applied to arithmetic operations on fuzzy intervals.

Fuzzy Arithmetic

Fuzzy Sets on the Real Numbers

There are many different types of fuzzy sets.

Very interesting are fuzzy sets defined on set $\mathbb{R}$ of real numbers.

Membership functions of such sets, *i.e.*

$$\mu : \mathbb{R} \to [0, 1],$$

clearly indicate quantitative meaning.

Such concepts may essentially characterize states of fuzzy variables.

They play important role in many applications, *e.g.* fuzzy control, decision making, approximate reasoning, optimization, and statistics with imprecise probabilities.

Some Special Fuzzy Sets I

Here, we only consider special classes $\mathcal{F}(\mathbb{R})$ of fuzzy sets μ on $\mathbb{R}$.

Definition

(a) $\mathcal{F}_N(\mathbb{R}) \stackrel{\text{def}}{=} \{\mu \in \mathcal{F}(\mathbb{R}) \mid \exists x \in \mathbb{R} : \mu(x) = 1\}$,

(b) $\mathcal{F}_C(\mathbb{R}) \stackrel{\text{def}}{=} \{\mu \in \mathcal{F}_N(\mathbb{R}) \mid \forall \alpha \in (0,1] : [\mu]_\alpha \text{ is compact } \}$,

(c) $\mathcal{F}_I(\mathbb{R}) \stackrel{\text{def}}{=} \{\mu \in \mathcal{F}_N(\mathbb{R}) \mid \forall a, b, c \in \mathbb{R} : c \in [a, b] \Rightarrow \mu(c) \geq \min\{\mu(a), \mu(b)\} \}$.

Some Special Fuzzy Sets II

An element in $\mathcal{F}_N(\mathbb{R})$ is called **normal fuzzy set**:

- It's meaningful if $\mu \in \mathcal{F}_N(\mathbb{R})$ is used as *imprecise description* of an existing (but not precisely measurable) variable $\subseteq \mathbb{R}$.
- In such cases it would not be plausible to assign maximum membership degree of 1 to no single real number.

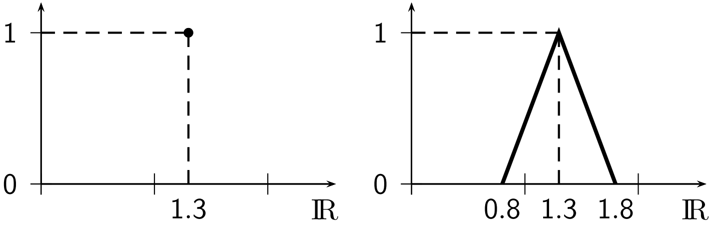
Sets in $\mathcal{F}_C(\mathbb{R})$ are **upper semi-continuous**:

- Function f is upper semi-continuous at point x_0 if values near x_0 are either close to $f(x_0)$ or less than $f(x_0)$
  $\Rightarrow \lim_{x \to x_0} \sup f(x) \leq f(x_0)$.
- This simplifies arithmetic operations applied to them.

Fuzzy sets in $\mathcal{F}_I(\mathbb{R})$ are called **fuzzy intervals**:

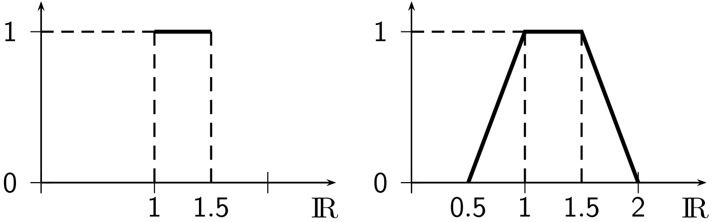
- The are *normal* and *fuzzy convex*.
- Their core is a classical interval.
- $\mu \in \mathcal{F}_I(\mathbb{R})$ for real numbers are called **fuzzy numbers**.

Comparison of Crisp Sets and Fuzzy Sets on $\mathbb{R}$



"exactly 1.3"

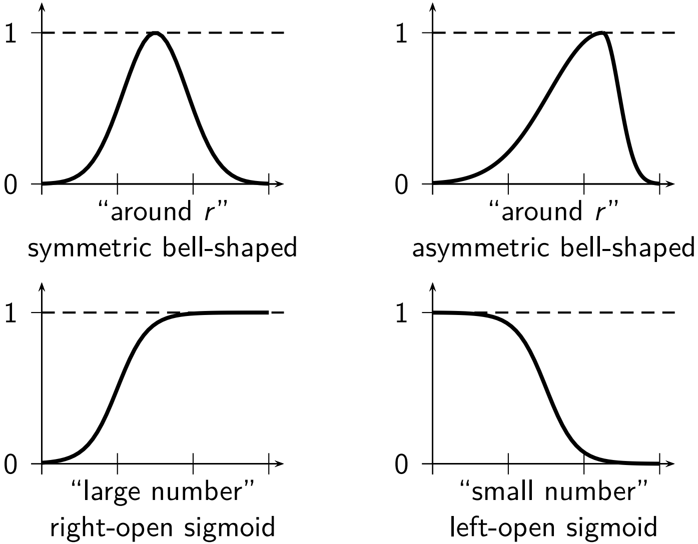
"close to 1.3"



crisp interval

fuzzy interval

Basic Types of Fuzzy Numbers



Quantitative Fuzzy Variables

The concept of a fuzzy number plays fundamental role in formulating *quantitative fuzzy variables*.

These are variables whose states are fuzzy numbers.

When the fuzzy numbers represent linguistic concepts, *e.g.*

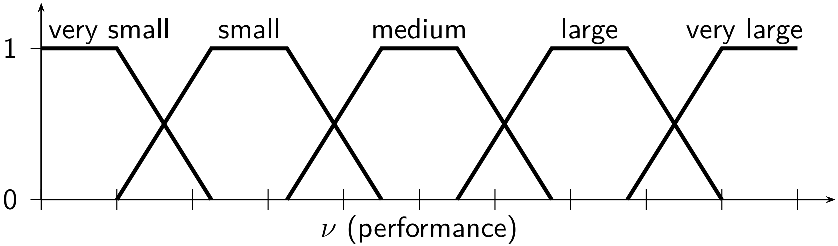
very small, *small*, *medium*, etc.

then final constructs are called **linguistic variables**.

Each linguistic variable is defined in terms of *base variable* which is a variable in classical sense, *e.g.* temperature, pressure, age.

Linguistic terms representing approximate values of base variable are captured by appropriate fuzzy numbers.

Linguistic Variables



Each linguistic variable is defined by quintuple (ν, T, X, g, m).

- *name* ν of the variable
- set T of *linguistic terms* of ν
- *base variable* $X \subseteq \mathbb{R}$
- *syntactic rule* g (grammar) for generating linguistic terms
- *semantic rule* m that assigns *meaning* $m(t)$ to every $t \in T$, *i.e.* $m : T \to \mathcal{F}(X)$

Operations on Linguistic Variables

To deal with linguistic variables, consider

- not only set-theoretic operations
- but also arithmetic operations on fuzzy numbers (*i.e.* interval arithmetic).

e.g. statistics:

- Given a sample = (*small*, *medium*, *small*, *large*, ...).
- How to define mean value or standard deviation?

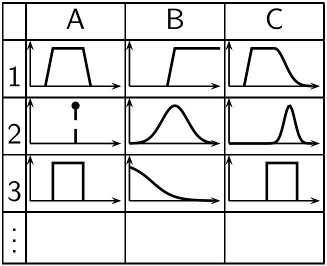
Analysis of Linguistic Data

Linguistic Data

	A	B	C
1	large	very large	medium
2	2.5	medium	about 7
3	[3, 4]	small	[7, 8]
⋮			

linguistic modeling →

Fuzzy Data



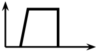
↓ *computing with words*

↓ *statistics with fuzzy sets*

"The mean *w.r.t.* A is approximately 4."

← *linguistic approximation*

mean of attribute A



Example – Application of Linguistic Data

Consider the problem to model the climatic conditions of several towns.

A tourist may want information about tourist attractions.

Assume that linguistic random samples are based on subjective observations of selected people, *e.g.*

- climatic attribute *clouding*
- linguistic values *cloudless*, *clear*, *fair*, *cloudy*, ...

Example – Linguistic Modeling by an Expert

The attribute *clouding* is modeled by elementary linguistic values, *e.g.*

$$
\begin{aligned}
\text{cloudless} &\mapsto \text{sigmoid}(0, -0.07)\\
\text{clear} &\mapsto \text{Gauss}(25, 15)\\
\text{fair} &\mapsto \text{Gauss}(50, 20)\\
\text{cloudy} &\mapsto \text{Gauss}(75, 15)\\
\text{overcast} &\mapsto \text{sigmoid}(100, 0.07)\\
\text{exactly})(x) &\mapsto \text{exact}(x)\\
\text{approx})(x) &\mapsto \text{Gauss}(x, 3)\\
\text{between}(x, y) &\mapsto \text{rectangle}(x, y)\\
\text{approx\_between}(x, y) &\mapsto \text{trapezoid}(x - 20, x, y, y + 20)
\end{aligned}
$$

where $x, y \in [0, 100] \subset \mathbb{R}$.

Example

Gauss(a, b) is, *e.g.* a function defined by

$$f(x) = \exp\left(-\left(\frac{x-a}{b}\right)^2\right), \qquad x, a, b \in \mathbb{R}, \quad b > 0$$

induced language of expressions:

<expression> := <elementary linguistic value> |
(<expression>) |
{ not | dil | con | int } <expression> |
<expression> { and | or } <expression> ,

e.g. approx(x) *and cloudy* is represented by function

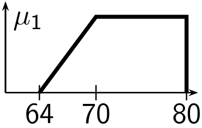
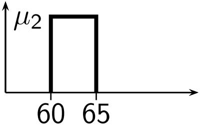
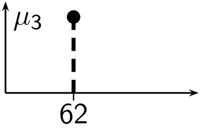
$$\min\{\text{Gauss}(x, 3), \text{Gauss}(75, 15)\}\,.$$

Example – Linguistic Random Sample

Attribute	:	Clouding
Observations	:	Limassol, Cyprus
2009/10/23	:	cloudy
2009/10/24	:	dil approx_between(50, 70)
2009/10/25	:	fair or cloudy
2009/10/26	:	approx(75)
2009/10/27	:	dil(clear or fair)
2009/10/28	:	int cloudy
2009/10/29	:	con fair
2009/11/30	:	approx(0)
2009/11/31	:	cloudless
2009/11/01	:	cloudless or dil clear
2009/11/02	:	overcast
2009/11/03	:	cloudy and between(70, 80)
...	:	...
2009/11/10	:	clear

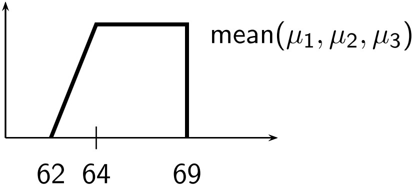
Statistics with fuzzy sets are necessary to analyze linguistic data.

Example – Ling. Random Sample of 3 People

no.	age (linguistic data)	age (fuzzy data)
1	approx. between 70 and 80 and definitely not older than 80	
2	between 60 and 65	
3	62	

Example – Mean Value of Ling. Random Sample

$$\text{mean}(\mu_1, \mu_2, \mu_3) = \frac{1}{3}(\mu_1 \oplus \mu_2 \oplus \mu_3)$$



i.e. approximately between 64 and 69 but not older than 69

Efficient Operations I

How to define arithmetic operations for calculating with $\mathcal{F}(\mathbb{R})$?

Using extension principle for sum $\mu \oplus \mu'$, product $\mu \odot \mu'$ and reciprocal value $\mathrm{rec}(\mu)$ of arbitrary fuzzy sets $\mu, \mu' \in \mathcal{F}(\mathbb{R})$

$$\begin{aligned}
(\mu \oplus \mu')(t) &= \sup\left\{\min\{\mu(x_1), \mu'(x_2)\} \mid x_1, x_2 \in \mathbb{R}, x_1 + x_2 = t\right\},\\
(\mu \odot \mu')(t) &= \sup\left\{\min\{\mu(x_1), \mu'(x_2)\} \mid x_1, x_2 \in \mathbb{R}, x_1 \cdot x_2 = t\right\},\\
\mathrm{rec}(\mu)(t) &= \sup\left\{\mu(x) \;\middle|\; x \in \mathbb{R} \setminus \{0\}, \frac{1}{x} = t\right\}.
\end{aligned}$$

In general, operations on fuzzy sets are much more complicated (especially if vertical instead of horizontal representation is applied).

It's desirable to reduce fuzzy arithmetic to ordinary set arithmetic.

Then, we apply elementary operations of *interval arithmetic*.

Efficient Operations II

Definition
A family $(A_\alpha)_{\alpha\in(0,1)}$ of sets is called *set representation* of $\mu \in \mathcal{F}_N(\mathbb{R})$ if

(a) $0 < \alpha < \beta < 1 \Longrightarrow A_\beta \subseteq A_\alpha \subseteq \mathbb{R}$ and

(b) $\mu(t) = \sup\{\alpha \in [0,1] \mid t \in A_\alpha\}$

holds where $\sup \emptyset = 0$.

Theorem
Let $\mu \in F_N(\mathbb{R})$. The family $(A_\alpha)_{\alpha\in(0,1)}$ of sets is a set representation of μ if and only if

$$[\mu]_{\underline{\alpha}} = \{t \in \mathbb{R} \mid \mu(t) > \alpha\} \subseteq A_\alpha \subseteq \{t \in \mathbb{R} \mid \mu(t) \geq \alpha\} = [\mu]_\alpha$$

is valid for all $\alpha \in (0,1)$.

Efficient Operations III

Theorem

Let $\mu_1, \mu_2, \ldots, \mu_n$ be normal fuzzy sets of $\mathbb{R}$ and $\phi : \mathbb{R}^n \to \mathbb{R}$ be a mapping. Then the following holds:

(a) $\forall \alpha \in [0,1) : [\hat{\phi}(\mu_1, \ldots, \mu_n)]_{\underline{\alpha}} = \phi([\mu_1]_{\underline{\alpha}}, \ldots, [\mu_n]_{\underline{\alpha}})$,

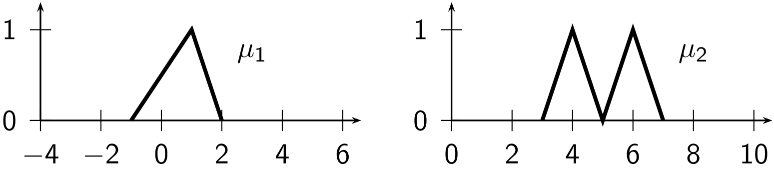
(b) $\forall \alpha \in (0,1] : [\hat{\phi}(\mu_1, \ldots, \mu_n)]_{\alpha} \supseteq \phi([\mu_1]_{\alpha}, \ldots, [\mu_n]_{\alpha})$,

(c) if $((A_i)_\alpha)_{\alpha \in (0,1)}$ *is a set representation of* μ_i *for* $1 \leq i \leq n$, *then* $(\phi((A_1)_\alpha, \ldots, (A_n)_\alpha))_{\alpha \in (0,1)}$ *is a set representation of* $\hat{\phi}(\mu_1, \ldots, \mu_n)$.

For arbitrary mapping ϕ, set representation of its extension $\hat{\phi}$ can be obtained with help of set representation $((A_i)_\alpha)_{\alpha \in (0,1)}, i = 1, 2, \ldots, n$.

It's used to carry out arithmetic operations on fuzzy sets efficiently.

Example I



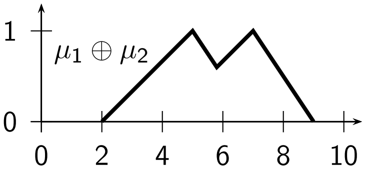
For μ_1, μ_2, the set representations are

- $[\mu_1]_\alpha = [2\alpha - 1, 2 - \alpha]$,
- $[\mu_2]_\alpha = [\alpha + 3, 5 - \alpha] \cup [\alpha + 5, 7 - \alpha]$.

Let $\text{add}(x, y) = x + y$, then $(A_\alpha)_{\alpha \in (0,1)}$ represents $\mu_1 \oplus \mu_2$

$$A_\alpha = \text{add}([\mu_1]_\alpha, [\mu_2]_\alpha) = [3\alpha + 2, 7 - 2\alpha] \cup [3\alpha + 4, 9 - 2\alpha]$$
$$= \begin{cases} [3\alpha + 2, 7 - 2\alpha] \cup [3\alpha + 4, 9 - 2\alpha], & \text{if } \alpha \geq 0.6 \\ [3\alpha + 2, 9 - 2\alpha], & \text{if } \alpha \leq 0.6. \end{cases}$$

Example II



$$(\mu_1 \oplus \mu_2)(x) = \begin{cases} \frac{x-2}{3}, & \text{if } 2 \leq x \leq 5 \\ \frac{7-x}{2}, & \text{if } 5 \leq x \leq 5.8 \\ \frac{x-4}{3}, & \text{if } 5.8 \leq x \leq 7 \\ \frac{9-x}{2}, & \text{if } 7 \leq x \leq 9 \\ 0, & \text{otherwise} \end{cases}$$

Interval Arithmetic I

Determining the set representations of arbitrary combinations of fuzzy sets can be reduced very often to simple interval arithmetic.

Using fundamental operations of arithmetic leads to the following ($a, b, c, d \in \mathbb{R}$):

$$[a, b] + [c, d] = [a + c, b + d]$$
$$[a, b] - [c, d] = [a - d, b - c]$$

$$[a, b] \cdot [c, d] = \begin{cases} [ac, bd], & \text{for } a \geq 0 \land c \geq 0 \\ [bd, ac], & \text{for } b < 0 \land d < 0 \\ [\min\{ad, bc\}, \max\{ad, bc\}], & \text{for } ab \geq 0 \land cd \geq 0 \land ac < 0 \\ [\min\{ad, bc\}, \max\{ac, bd\}], & \text{for } ab < 0 \lor cd < 0 \end{cases}$$

$$\frac{1}{[a, b]} = \begin{cases} \left[\frac{1}{b}, \frac{1}{a}\right], & \text{if } 0 \notin [a, b] \\ \left[\frac{1}{b}, \infty\right) \cup \left(-\infty, \frac{1}{a}\right], & \text{if } a < 0 \land b > 0 \\ \left[\frac{1}{b}, \infty\right), & \text{if } a = 0 \land b > 0 \\ \left(-\infty, \frac{1}{a}\right], & \text{if } a < 0 \land b = 0 \end{cases}$$

Interval Arithmetic II

In general, set representation of α-cuts of extensions $\hat{\phi}(\mu_1, \dots, \mu_n)$ cannot be determined directly from α-cuts.

It only works always for continuous ϕ and fuzzy sets in $\mathcal{F}_C(\mathbb{R})$.

Theorem

Let $\mu_1, \mu_2, \dots, \mu_n \in \mathcal{F}_C(\mathbb{R})$ *and* $\phi : \mathbb{R}^n \to \mathbb{R}$ *be a continuous mapping. Then*

$$\forall \alpha \in (0, 1] : [\hat{\phi}(\mu_1, \dots, \mu_n)]_\alpha = \phi([\mu_1]_\alpha, \dots, [\mu_n]_\alpha).$$

So, a horizontal representation is better than a vertical one.
Finding $\hat{\phi}$ values is easier than directly applying the extension principle.
However, all α-cuts cannot be stored in a computer.
Only a finite number of α-cuts can be stored.

Fuzzy Relations

Motivation

A **Crisp relation** represents presence or absence of association, interaction or interconnection between elements of ≥ 2 sets.

This concept can be generalized to various degrees or strengths of association or interaction between elements.

A **fuzzy relation** generalizes these degrees to membership grades.

So, a crisp relation is a restricted case of a fuzzy relation.

Definition of Relation

A **relation** among crisp sets $X_1, \dots, X_n$ is a subset of $X_1 \times \dots \times X_n$ denoted as $R(X_1, \dots, X_n)$ or $R(X_i \mid 1 \leq i \leq n)$.

So, the relation $R(X_1, \dots, X_n) \subseteq X_1 \times \dots \times X_n$ is set, too.

The basic concept of sets can be also applied to relations:

- containment, subset, union, intersection, complement

Each crisp relation can be defined by its characteristic function

$$R(x_1, \dots, x_n) = \begin{cases} 1, & \text{if and only if } (x_1, \dots, x_n) \in R, \\ 0, & \text{otherwise.} \end{cases}$$

The membership of $(x_1, \dots, x_n)$ in R signifies that the elements of $(x_1, \dots, x_n)$ are related to each other.

Relation as Ordered Set of Tuples

A relation can be written as a set of ordered tuples.

Thus $R(X_1, \ldots, X_n)$ represents n-dim. membership array $\boldsymbol{R} = [r_{i_1,\ldots,i_n}]$.

- Each element of i_1 of $\boldsymbol{R}$ corresponds to exactly one member of X_1.
- Each element of i_2 of $\boldsymbol{R}$ corresponds to exactly one member of X_2.
- And so on...

If $(x_1, \ldots, x_n) \in X_1 \times \ldots \times X_n$ corresponds to $r_{i_1,\ldots,i_n} \in \boldsymbol{R}$, then

$$r_{i_1,\ldots,i_n} = \begin{cases} 1, & \text{if and only if } (x_1, \ldots, x_n) \in R, \\ 0, & \text{otherwise.} \end{cases}$$

Fuzzy Relations

The characteristic function of a crisp relation can be generalized to allow tuples to have degrees of membership.

- Recall the generalization of the characteristic function of a crisp set!

Then a **fuzzy relation** is a fuzzy set defined on tuples $(x_1, \ldots, x_n)$ that may have varying degrees of membership within the relation.

The membership grade indicates strength of the present relation between elements of the tuple.

The fuzzy relation can also be represented by an n-dimensional membership array.

Example

Let R be a fuzzy relation between two sets $X = \{\text{New York City}, \text{Paris}\}$ and $Y = \{\text{Beijing}, \text{New York City}, \text{London}\}$.

R shall represent relational concept "very far".

It can be represented as two-dimensional membership array:

	NYC	Paris
Beijing	1	0.9
NYC	0	0.7
London	0.6	0.3

Cartesian Product of Fuzzy Sets: n Dimensions

Let $n \geq 2$ fuzzy sets $A_1, \ldots, A_n$ be defined in the universes of discourse $X_1, \ldots, X_n$, respectively.

The *Cartesian product* of $A_1, \ldots, A_n$ denoted by $A_1 \times \ldots \times A_n$ is a fuzzy relation in the product space $X_1 \times \ldots \times X_n$.

It is defined by its membership function

$$\mu_{A_1 \times \ldots \times A_n}(x_1, \ldots, x_n) = \top\left(\mu_{A_1}(x_1), \ldots, \mu_{A_n}(x_n)\right)$$

whereas $x_i \in X_i, \quad 1 \leq i \leq n$.

Usually $\top$ is the minimum (sometimes also the product).

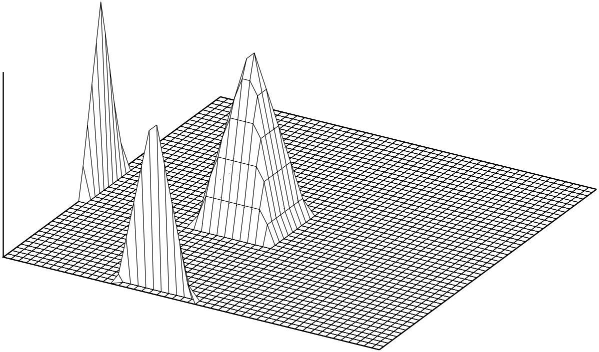
Cartesian Product of Fuzzy Sets: 2 Dimensions

A special case of the Cartesian product is when $n = 2$.

Then the Cartesian product of fuzzy sets $A \in \mathcal{F}(X)$ and $B \in \mathcal{F}(Y)$ is a fuzzy relation $A \times B \in \mathcal{F}(X \times Y)$ defined by

$$\mu_{A \times B}(x, y) = \top\left[\mu_A(x),\ \mu_B(y)\right], \quad \forall x \in X,\ \forall y \in Y.$$

Example: Cartesian Product in $\mathcal{F}(X \times Y)$ with t-norm = min



Subsequences

Consider the Cartesian product of all sets in the family

$$\mathcal{X} = \{X_i \mid i \in \mathbb{N}_n = \{1, 2, \ldots, n\}\}.$$

For each sequence (n-tuple) $\boldsymbol{x} = (x_1, \ldots, x_n) \in \times_{i \in \mathbb{N}_n} X_i$
and each sequence (r-tuple, $r \leq n$) $\boldsymbol{y} = (y_1, \ldots, y_r) \in \times_{j \in J} X_j$
where $J \subseteq \mathbb{N}_n$ and $|J| = r$
$\boldsymbol{y}$ is called **subsequence** of $\boldsymbol{x}$ if and only if $y_j = x_j, \ \forall j \in J$.

$\boldsymbol{y} \prec \boldsymbol{x}$ denotes that $\boldsymbol{y}$ is subsequence of $\boldsymbol{x}$.

Projection

Given a relation $R(x_1, \ldots, x_n)$.

Let $[R \downarrow \mathcal{Y}]$ denote the **projection** of R on $\mathcal{Y}$.

It disregards all sets in X except those in the family

$$\mathcal{Y} = \{X_j \mid j \in J \subseteq \mathbb{N}_n\}.$$

Then $[R \downarrow \mathcal{Y}]$ is a fuzzy relation whose membership function is defined on the Cartesian product of the sets in $\mathcal{Y}$

$$[R \downarrow \mathcal{Y}](\boldsymbol{y}) = \max_{\boldsymbol{x} \succ \boldsymbol{y}} R(\boldsymbol{x}).$$

Under special circumstances, this projection can be generalized by replacing the max operator by another t-conorm.

Example

Consider the sets $X_1 = \{0,1\}$, $X_2 = \{0,1\}$, $X_3 = \{0,1,2\}$ and the ternary fuzzy relation on $X_1 \times X_2 \times X_3$ defined as follows:

Let $R_{ij} = [R \downarrow \{X_i, X_j\}]$ and $R_i = [R \downarrow \{X_i\}]$ for all $i, j \in \{1,2,3\}$.

Using this notation, all possible projections of R are given below.

$(x_1,$	$x_2,$	$x_3)$	$R(x_1,x_2,x_3)$	$R_{12}(x_1,x_2)$	$R_{13}(x_1,x_3)$	$R_{23}(x_2,x_3)$	$R_1(x_1)$	$R_2(x_2)$	$R_3(x_3)$
0	0	0	0.4	0.9	1.0	0.5	1.0	0.9	1.0
0	0	1	0.9	0.9	0.9	0.9	1.0	0.9	0.9
0	0	2	0.2	0.9	0.8	0.2	1.0	0.9	1.0
0	1	0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
0	1	1	0.0	1.0	0.9	0.5	1.0	1.0	0.9
0	1	2	0.8	1.0	0.8	1.0	1.0	1.0	1.0
1	0	0	0.5	0.5	0.5	0.5	1.0	0.9	1.0
1	0	1	0.3	0.5	0.5	0.9	1.0	0.9	0.9
1	0	2	0.1	0.5	1.0	0.2	1.0	0.9	1.0
1	1	0	0.0	1.0	0.5	1.0	1.0	1.0	1.0
1	1	1	0.5	1.0	0.5	0.5	1.0	1.0	0.9
1	1	2	1.0	1.0	1.0	1.0	1.0	1.0	1.0

Example: Detailed Calculation

Here, only consider the projection R_{12}:

$(x_1,$	$x_2,$	$x_3)$	$R(x_1, x_2, x_3)$	$R_{12}(x_1, x_2)$
0	0	0	0.4	
0	0	1	0.9	$\max[R(0,0,0), R(0,0,1), R(0,0,2)] = 0.9$
0	0	2	0.2	
0	1	0	1.0	
0	1	1	0.0	$\max[R(0,1,0), R(0,1,1), R(0,1,2)] = 1.0$
0	1	2	0.8	
1	0	0	0.5	
1	0	1	0.3	$\max[R(1,0,0), R(1,0,1), R(1,0,2)] = 0.5$
1	0	2	0.1	
1	1	0	0.0	
1	1	1	0.5	$\max[R(1,1,0), R(1,1,1), R(1,1,2)] = 1.0$
1	1	2	1.0	

Cylindric Extension

Another operation on relations is called **cylindric extension**.

Let $\mathcal{X}$ and $\mathcal{Y}$ denote the same families of sets as used for projection.

Let R be a relation defined on Cartesian product of sets in family $\mathcal{Y}$.

Let $[R \uparrow \mathcal{X} \setminus \mathcal{Y}]$ denote the cylindric extension of R into sets X_1, $(i \in \mathbb{N}_n)$ which are in $\mathcal{X}$ but not in $\mathcal{Y}$.

It follows that for each $\boldsymbol{x}$ with $\boldsymbol{x} \succ \boldsymbol{y}$

$$[R \uparrow \mathcal{X} \setminus \mathcal{Y}](\boldsymbol{x}) = R(\boldsymbol{y}).$$

The cylindric extension

- produces **largest** fuzzy relation that is compatible with projection,
- is the least specific of all relations compatible with projection,
- guarantees that no information not included in projection is used to determine extended relation.

Example

Consider again the example for the projection.

The membership functions of the cylindric extensions of all projections are already shown in the table under the assumption that their arguments are extended to (x_1, x_2, x_3) *e.g.*

$$[R_{23} \uparrow \{X_1\}](0,0,2) = [R_{23} \uparrow \{X_1\}](1,0,2) = R_{23}(0,2) = 0.2.$$

In this example none of the cylindric extensions are equal to the original fuzzy relation.

This is identical with the respective projections.

Some information was lost when the given relation was replaced by any one of its projections.

Cylindric Closure

Relations that can be reconstructed from one of their projections by cylindric extension exist.

However, they are rather rare.

It is more common that relation can be exactly reconstructed

- from several of its projections (max),
- by taking set intersection of their cylindric extensions (min).

The resulting relation is usually called **cylindric closure**.

Let the set of projections $\{P_i \mid i \in I\}$ of a relation on $\mathcal{X}$ be given.

Then the cylindric closure $\text{cyl}\{P_i\}$ is defined for each $\boldsymbol{x} \in \mathcal{X}$ as

$$\text{cyl}\{P_i\}(\boldsymbol{x}) = \min_{i \in I}[P_i \uparrow \mathcal{X} \setminus \mathcal{Y}_i](\boldsymbol{x}).$$

$\mathcal{Y}_i$ denotes the family of sets on which P_i is defined.

Example

Consider again the example for the projection.
The cylindric closures of three families of the projections are shown below:

$(x_1,$	$x_2,$	$x_3)$	$R(x_1, x_2, x_3)$	$\mathrm{cyl}(R_{12}, R_{13}, R_{23})$	$\mathrm{cyl}(R_1, R_2, R_3)$	$\mathrm{cyl}(R_{12}, R_3)$
0	0	0	0.4	0.5	0.9	0.9
0	0	1	0.9	0.9	0.9	0.9
0	0	2	0.2	0.2	0.9	0.9
0	1	0	1.0	1.0	1.0	1.0
0	1	1	0.0	0.5	0.9	0.9
0	1	2	0.8	0.8	1.0	1.0
1	0	0	0.5	0.5	0.9	0.5
1	0	1	0.3	0.5	0.9	0.5
1	0	2	0.1	0.2	0.9	0.5
1	1	0	0.0	0.5	1.0	1.0
1	1	1	0.5	0.5	0.9	0.9
1	1	2	1.0	1.0	1.0	1.0

None of them is the same as the original relation R.

So the relation R is not fully reconstructable from its projections.

Binary Fuzzy Relations

Motivation and Domain

Binary relations are significant among n-dimensional relations.

They are (in some sense) generalized mathematical functions.

On the contrary to functions from X to Y, binary relations $R(X, Y)$ may assign to each element of X two or more elements of Y.

Some basic operations on functions, *e.g.* inverse and composition, are applicable to binary relations as well.

Given a fuzzy relation $R(X, Y)$.

Its **domain** $\operatorname{dom} R$ is the fuzzy set on X whose membership function is defined for each $x \in X$ as

$$\operatorname{dom} R(x) = \max_{y \in Y}\{R(x, y)\},$$

i.e. each element of X belongs to the domain of R to a degree equal to the strength of its strongest relation to any $y \in Y$.

Range and Height

The **range** ran of $R(X, Y)$ is a fuzzy relation on Y whose membership function is defined for each $y \in Y$ as

$$\operatorname{ran} R(y) = \max_{x \in X}\{R(x, y)\},$$

i.e. the strength of the strongest relation which each $y \in Y$ has to an $x \in X$ equals to the degree of membership of y in the range of R.

The **height** h of $R(X, Y)$ is a number defined by

$$h(R) = \max_{y \in Y} \max_{x \in X}\{R(x, y)\}.$$

$h(R)$ is the largest membership grade obtained by any pair $(x, y) \in R$.

Representation and Inverse

Consider *e.g.* the **membership matrix** $\boldsymbol{R} = [r_{xy}]$ with $r_{xy} = R(x, y)$.

Its **inverse** $R^{-1}(Y, X)$ of $R(X, Y)$ is a relation on $Y \times X$ defined by

$$R^{-1}(y, x) = R(x, y), \quad \forall x \in X, \ \forall y \in Y.$$

$\boldsymbol{R}^{-1} = [r_{xy}^{-1}]$ representing $R^{-1}(y, x)$ is the transpose of $\boldsymbol{R}$ for $R(X, Y)$

$$(\boldsymbol{R}^{-1})^{-1} = \boldsymbol{R}$$

Standard Composition

Consider the binary relations $P(X, Y)$, $Q(Y, Z)$ with common set Y.

The **standard composition** of P and Q is defined as

$$(x, z) \in P \circ Q \iff \exists y \in Y : \{(x, y) \in P \wedge (y, z) \in Q\}.$$

In the fuzzy case this is generalized by

$$[P \circ Q](x, z) = \sup_{y \in Y} \min\{P(x, y),\ Q(y, z)\},\ \ \forall x \in X,\ \forall z \in Z.$$

If Y is finite, sup operator is replaced by max.

Then the standard composition is also called **max-min composition**.

Inverse of Standard Composition

The inverse of the max-min composition follows from its definition:

$$[P(X,Y) \circ Q(Y,Z)]^{-1} = Q^{-1}(Z,Y) \circ P^{-1}(Y,X).$$

Its associativity also comes directly from its definition:

$$[P(X,Y)] \circ Q(Y,Z)] \circ R(Z,W) = P(X,Y) \circ [Q(Y,Z) \circ R(Z,W)].$$

Note that the standard composition is not commutative.

Matrix notation: $[r_{ij}] = [p_{ik}] \circ [q_{kj}]$ with $r_{ij} = \max_k \min(p_{ik}, q_{kj})$.

Example

$$\boldsymbol{P} \circ \boldsymbol{Q} = \boldsymbol{R}$$

$$\begin{bmatrix} .3 & .5 & .8 \\ 0 & .7 & 1 \\ .4 & .6 & .5 \end{bmatrix} \circ \begin{bmatrix} .9 & .5 & .7 & .7 \\ .3 & .2 & 0 & .9 \\ 1 & 0 & .5 & .5 \end{bmatrix} = \begin{bmatrix} .8 & .3 & .5 & .5 \\ 1 & .2 & 5 & .7 \\ .5 & .4 & .5 & .5 \end{bmatrix}$$

For instance:

$$\begin{aligned} r_{11} &= \max\{\min(p_{11}, q_{11}), \min(p_{12}, q_{21}), \min(p_{13}, q_{31})\} \\ &= \max\{\min(.3, .9), \min(.5, .3), \min(.8, 1)\} \\ &= .8 \\ r_{32} &= \max\{\min(p_{31}, q_{12}), \min(p_{32}, q_{22}), \min(p_{33}, q_{32})\} \\ &= \max\{\min(.4, .5), \min(.6, .2), \min(.5, 0)\} \\ &= .4 \end{aligned}$$

Example: Properties of Airplanes (Speed, Height, Type)

4 possible speeds:	s_1, s_2, s_3, s_4
3 heights:	h_1, h_2, h_3
2 types:	t_1, t_2

Consider the following fuzzy relations for airplanes:

- relation A between maximal speed and maximal height,
- relation B between maximal height and the type.

$\boldsymbol{A}$	h_1	h_2	h_3
s_1	1	.2	0
s_2	.1	1	0
s_3	0	1	1
s_4	0	.3	1

$\boldsymbol{B}$	t_1	t_2
h_1	1	0
h_2	.9	1
h_3	0	.9

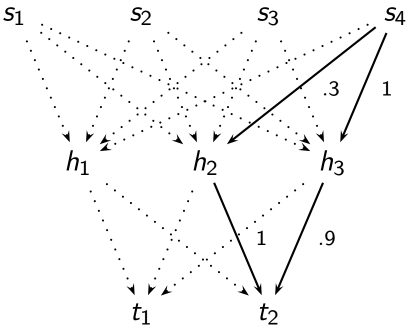
Example (cont.)

matrix multiplication scheme

			1	0
A	$\circ$	***B***	.9	1
			0	.9
1	.2	0	1	.2
.1	1	0	.9	1
0	1	1	.9	1
0	.3	1	.3	.9

$A \circ B$ speed-type relation

flow scheme



$$(A \circ B)(s_4, t_2) = \max\{\min\{.3,\ 1\},\ \min\{1,\ .9\}\}$$
$$= .9$$

Relational Join

A similar operation on two binary relations is the **relational join**.

It yields triples (whereas composition returned pairs).

For $P(X, Y)$ and $Q(Y, Z)$, the relational join $P * Q$ is defined by

$$[P * Q](x, y, z) = \min\{P(x, y),\ Q(y, z)\}, \quad \forall x \in X, \forall y \in Y, \forall z \in Z.$$

Then the max-min composition is obtained by aggregating the join by the maximum:

$$[P \circ Q](x, z) = \max_{y \in Y}[P * Q](x, y, z), \quad \forall x \in X, \forall z \in Z.$$

Example

The join $S = P * Q$ of the relations P and Q has the following membership function (shown below on left-hand side).

To convert this join into its corresponding composition $R = P \circ Q$ (shown on right-hand side),
The two indicated pairs of $S(x, y, z)$ are aggregated using max.

x	y	z	$\mu_S x, y, z$
1	a	α	.6
1	a	β	.7*
1	b	β	.5*
2	a	α	.6
2	a	β	.8
3	b	β	1
4	b	β	.4*
4	c	β	.3*

x	z	$\mu_R(x, z)$
1	α	.6
1	β	.7
2	α	.6
2	β	.8
3	β	.1
4	β	.4

For instance,

$$R(1, \beta) = \max\{S(1, a, \beta),\ S(1, b, \beta)\}$$
$$= \max\{.7,\ .5\} = .7$$

Binary Relations on a Single Set

Binary Relations on a Single Set

It is also possible to define crisp or fuzzy binary relations among elements of a single set X.

Such a binary relation can be denoted by $R(X, X)$ or $R(X^2)$ which is a subset of $X \times X = X^2$.

These relations are often referred to as **directed graphs** which is also an representation of them.

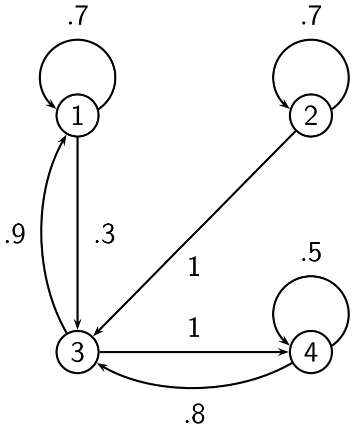
- Each element of X is represented as node.
- Directed connections between nodes indicate pairs of $x \in X$ for which the grade of the membership is nonzero.
- Each connection is labeled by its actual membership grade of the corresponding pair in R.

Example

An example of $R(X, X)$ defined on $X = \{1, 2, 3, 4\}$.

Two different representation are shown below.

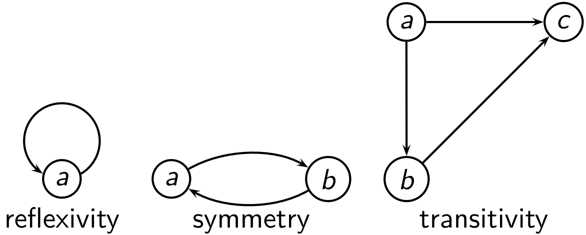
	1	2	3	4
1	.7	0	.3	0
2	0	.7	1	0
3	.9	0	0	1
3	0	0	.8	.5



Properties of Crisp Relations

A crisp relation $R(X, X)$ is called

- *reflexive* if and only if $\forall x \in X : (x, x) \in R$,
- *symmetric* if and only if $\forall x, y \in X : (x, y) \in R \leftrightarrow (y, x) \in R$,
- *transitive* if and only if $(x, z) \in R$ whenever both $(x, y) \in R$ and $(y, z) \in R$ for at least one $y \in X$.



All these properties are preserved under inversion of the relation.

Properties of Fuzzy Relations

These properties can be extended for fuzzy relations.

So one can define them in terms of the membership function of the relation.

A fuzzy relation $R(X, X)$ is called

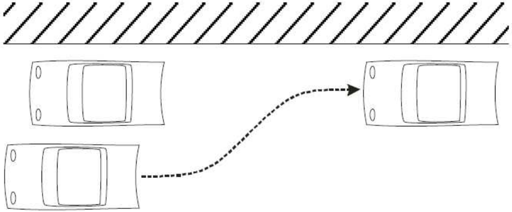
- *reflexive* if and only if $\forall x \in X : R(x, x) = 1$,
- *symmetric* if and only if $\forall x, y \in X : R(x, y) = R(y, x)$,
- *transitive* if it satisfies

$$R(x, z) \geq \max_{y \in Y} \min\{R(x, y),\ R(y, z)\}, \quad \forall (x, z) \in X^2.$$

Note that a fuzzy binary relation that is reflexive, symmetric and transitive is called **fuzzy equivalence relation**.

Fuzzy Control Basics

Example – Parking a car backwards



Questions:

What is the meaning of satisfactory parking?

Demand on precision?

Realization of control?

Fuzzy Control

Biggest success of fuzzy systems in industry and commerce.

Special kind of non-linear table-based control method.

Definition of non-linear transition function can be made without specifying each entry individually.

Examples: technical systems

- Electrical engine moving an elevator,
- Heating installation

Goal: define certain behavior

- Engine should maintain certain number of revolutions per minute.
- Heating should guarantee certain room temperature.

Table-based Control

Control systems all share a time-dependent **output variable**:

- Revolutions per minute,
- Room temperature.

Output is controlled by **control variable**:

- Adjustment of current,
- Thermostat.

Also, **disturbance variables** influence output:

- Load of elevator, . . . ,
- Outside temperature or sunshine through a window, . . .

Table-based Control

Computation of actual value incorporates both

control variable measurements of current output variable ξ and

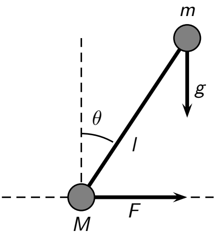
change of output variable $\Delta\xi = \frac{d\xi}{dt}$.

If ξ is given in finite time intervals,

then set $\Delta\xi(t_{n+1}) = \xi(t_{n+1}) - \xi(t_n)$.

In this case measurement of $\Delta\xi$ not necessary.

Example: Cartpole Problem



Balance an upright standing pole by moving its foot.

Lower end of pole can be moved unrestrained along horizontal axis.

Mass *m* at foot and mass *M* at head.

Influence of mass of shaft itself is negligible.

Determine force *F* (control variable) that is necessary to balance pole standing upright.

That is measurement of following output variables:

- angle θ of pole in relation to vertical axis,
- change of angle, *i.e.* triangular velocity $\dot{\theta} = \frac{d\theta}{dt}$.

Both should converge to zero.

Notation

Input variables $\xi_1, \ldots, \xi_n$, control variable η

Measurements: used to determine actual value of η

η may specify change of η.

Assumption: ξ_i, $1 \leq i \leq n$ is value of X_i, $\eta \in Y$

Solution: **control function** φ

$$\varphi : X_1 \times \ldots \times X_n \to Y$$
$$(x_1, \ldots, x_n) \mapsto y$$

Example: Cartpole Problem (cont.)

Angle $\theta \in X_1 = [-90\,^\circ, 90\,^\circ]$

Theoretically, every angle velocity $\dot{\theta}$ possible.

Extreme $\dot{\theta}$ are artificially achievable.

Assume $-45\,^\circ/\mathrm{s} \leq \dot{\theta} \leq 45\,^\circ/\mathrm{s}$ holds,
i.e. $\dot{\theta} \in X_2 = [-45\,^\circ/\mathrm{s}, 45\,^\circ/\mathrm{s}]$.

Absolute value of force $|F| \leq 10\,\mathrm{N}$.

Thus define $F \in Y = [-10\,\mathrm{N}, 10\,\mathrm{N}]$.

Example: Cartpole Problem (cont.)

Differential equation of cartpole problem:

$$(M+m)\sin^2\theta\cdot l\cdot\ddot{\theta}+m\cdot l\cdot\sin\theta\cos\theta\cdot\dot{\theta}^2-(M+m)\cdot g\cdot\sin\theta=-F\cdot\cos\theta$$

Compute $F(t)$ such that $\theta(t)$ and $\dot{\theta}(t)$ converge towards zero quickly.

Physical analysis demands knowledge about physical process.

Problems of Classical Approach

Often very difficult or even impossible to specify accurate mathematical model.

Description with differential equations is very complex.

Profound physical knowledge from engineer.

Exact solution can be very difficult.

Should be possible: to control process without physical-mathematical model,
e.g. human being knows how to ride bike without knowing existence of differential equations.

Fuzzy Approach

Simulate behavior of human who knows how to control.

That is a **knowledge-based analysis**.

Directly ask expert to perform analysis.

Then expert specifies knowledge as **linguistic rules**, *e.g.* for cartpole problem:

"If θ is approximately zero and $\dot{\theta}$ is also approximately zero,
then F has to be approximately zero, too."

Fuzzy Approach: Fuzzy Partitioning

1. Formulate set of linguistic rules:

Determine linguistic terms (represented by fuzzy sets).

$X_1, \ldots, X_n$ and Y is partitioned into fuzzy sets.

Define p_1 distinct fuzzy sets $\mu_1^{(1)}, \ldots, \mu_{p_1}^{(1)} \in \mathcal{F}(X_1)$ on set X_1.

Associate linguistic term with each set.

Fuzzy Approach: Fuzzy Partitioning II

Of set X_1 corresponds to interval $[a, b]$ of real line, then $\mu_1^{(1)}, \ldots, \mu_{p_1}^{(1)} \in \mathcal{F}(X_1)$ are triangular functions

$$\begin{aligned} \mu_{x_0,\varepsilon} : [a, b] &\to [0, 1] \\ x &\mapsto 1 - \min\{\varepsilon \cdot |x - x_0|,\ 1\}. \end{aligned}$$

If $a < x_1 < \ldots < x_{p_1} < b$, only $\mu_2^{(1)}, \ldots, \mu_{p_1-1}^{(1)}$ are triangular.

Boundaries are treated differently.

Fuzzy Approach: Fuzzy Partitioning III

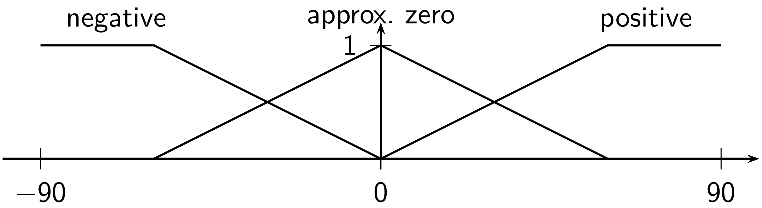
left fuzzy set:

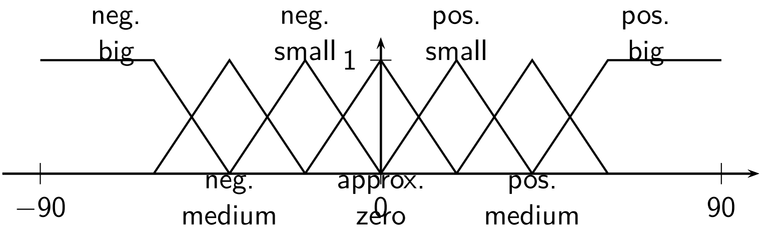
$$\mu_1^{(1)} : [a, b] \to [0, 1]$$
$$x \mapsto \begin{cases} 1, & \text{if } x \leq x_1 \\ 1 - \min\{\varepsilon \cdot (x - x_1),\ 1\} & \text{otherwise} \end{cases}$$

right fuzzy set:

$$\mu_{p_1}^{(1)} : [a, b] \to [0, 1]$$
$$x \mapsto \begin{cases} 1, & \text{if } x_{p_1} \leq x \\ 1 - \min\{\varepsilon \cdot (x_{p_1} - x),\ 1\} & \text{otherwise} \end{cases}$$

Coarse and Fine Fuzzy Partitions





Example: Cartpole Problem (cont.)

X_1 partitioned into 7 fuzzy sets.

Support of fuzzy sets: intervals with length $\frac{1}{4}$ of whole range X_1.

Similar fuzzy partitions for X_2 and Y.

Next step: specify rules

$$\text{if } \xi_1 \text{ is } A^{(1)} \text{ and } \ldots \text{ and } \xi_n \text{ is } A^{(n)} \text{ then } \eta \text{ is } B,$$

$A^{(1)}, \ldots, A^{(n)}$ and B represent linguistic terms corresponding to $\mu^{(1)}, \ldots, \mu^{(n)}$ and μ according to $X_1, \ldots, X_n$ and Y.

Rule base consists of k rules.

Example: Cartpole Problem (cont.)

$\dot{\theta}$ \ θ	nb	nm	ns	az	ps	pm	pb
nb			ps	pb			
nm				pm			
ns	nm		ns	ps			
az	nb	nm	ns	az	ps	pm	pb
ps				ns	ps		pm
pm				nm			
pb				nb	ns		

19 rules for cartpole problem, often not necessary to determine all table entries *e.g.*

If θ is *approximately zero* and $\dot{\theta}$ is *negative medium*
then F is *positive medium*.

Fuzzy Approach: Challenge

How to define function $\varphi : X \to Y$ that fits to rule set?

Idea:

Represent set of rules as fuzzy relation.

Specify desired table-based controller by this fuzzy relation.

Fuzzy Relation

Consider only crisp sets.

Then, solving control problem = specifying control function $\varphi : X \to Y$.

φ corresponds to relation

$$R_\varphi = \{(x, \varphi(x)) \mid x \in X\} \subseteq X \times Y.$$

For measured input $x \in X$, control value

$$\{\varphi(x)\} = \{x\} \circ R_\varphi.$$

Fuzzy Control Rules

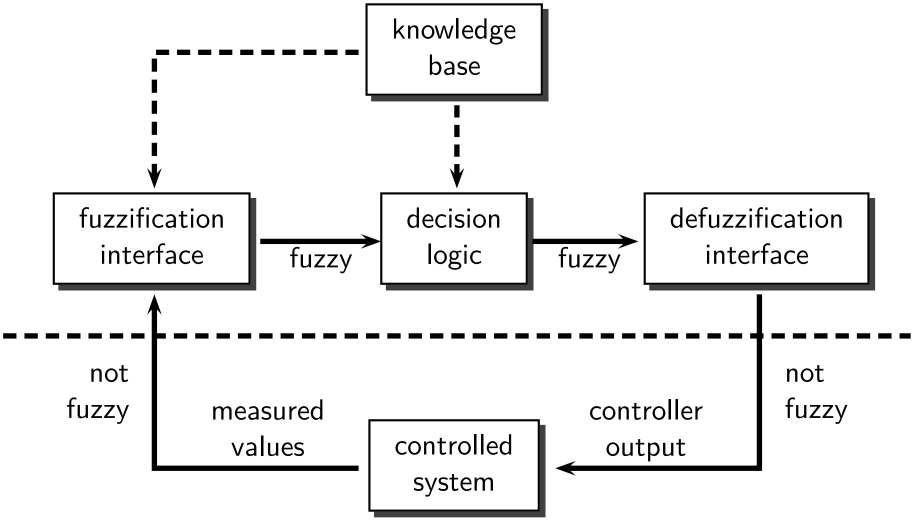
If temperature is very high **and** pressure is slightly low,

then heat change should be slightly negative.

If rate of descent = positive big **and** airspeed = negative big **and** glide slope = positive big,

then rpm change = positive big **and** elevator angle change = insignificant change.

Architecture of a Fuzzy Controller I



Architecture of a Fuzzy Controller II

Fuzzification interface

- receives current input value (eventually maps it to suitable domain),
- converts input value into linguistic term or into fuzzy set.

Knowledge base (consists of **data base** and **rule base**)

- Data base contains information about boundaries, possible domain transformations, and fuzzy sets with corresponding linguistic terms.
- Rule base contains linguistic control rules.

Decision logic (represents processing unit)

- computes output from measured input accord. to knowledge base.

Defuzzification interface (represents processing unit)

- determines crisp output value
  (and eventually maps it back to appropriate domain).

Fuzzy Rule Bases

Approximate Reasoning with Fuzzy Rules

General schema

Rule 1:	**if** X is M_1, **then** Y is N_1
Rule 2:	**if** X is M_2, **then** Y is N_2
$\vdots$	$\vdots$
Rule r:	**if** X is M_r, **then** Y is N_r
Fact:	X is M'
Conclusion:	Y is N'

Given r **if-then rules** and fact "X is M'", we conclude "Y is N'".

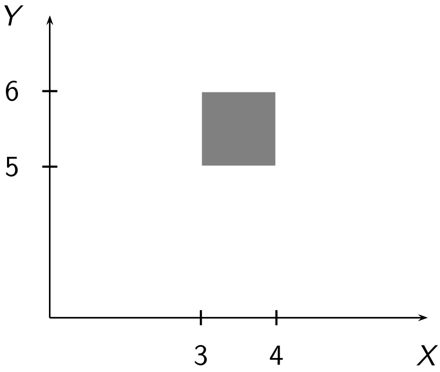
Typically used in **fuzzy controllers**.

Approximate Reasoning

Disjunctive Imprecise Rule

Imprecise rule: **if** $X = [3,4]$ **then** $Y = [5,6]$.

Interpretation: values coming from $[3,4] \times [5,6]$.

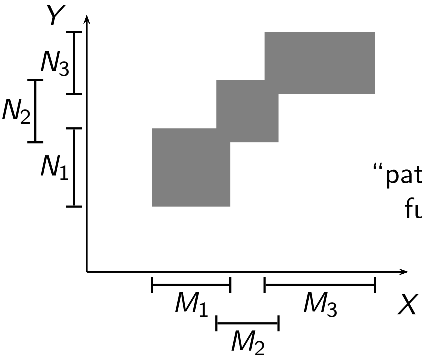


Approximate Reasoning

Disjunctive Imprecise Rules

Several imprecise rules: **if** $X = M_1$ **then** $Y = N_1$, **if** $X = M_2$ **then** $Y = N_2$, **if** $X = M_3$ **then** $Y = N_3$.

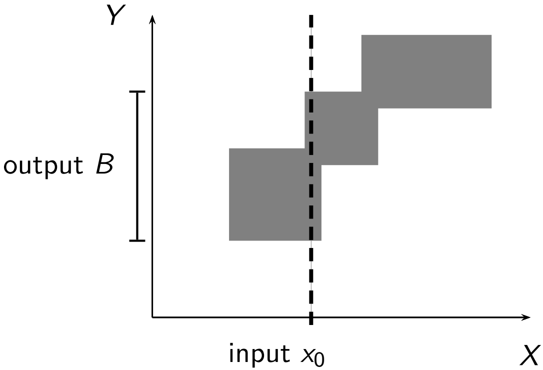
Interpretation: rule 1 as well as rule 2 as well as rule 3 hold true.



$$S = \bigcup_{i=1}^{r} M_i \times N_i$$

"patchwork rug" describing function's behavior as indicator function

Approximate Reasoning: Conclusion



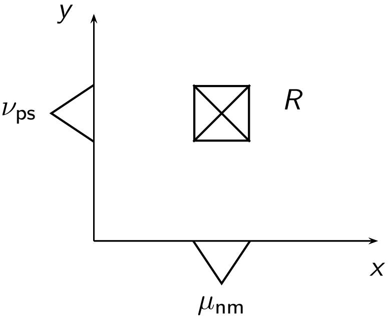
$$\{x_0\} \circ S = B$$

Approximate Reasoning

Disjunctive Fuzzy Rules

one fuzzy rule:

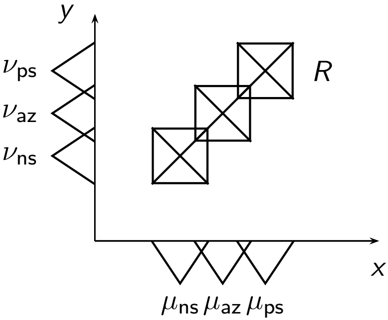
if $X = \text{nm}$ **then** $Y = \text{ps}$



$R = \mu_{\text{nm}} \times \nu_{\text{ps}}$

several fuzzy rules:

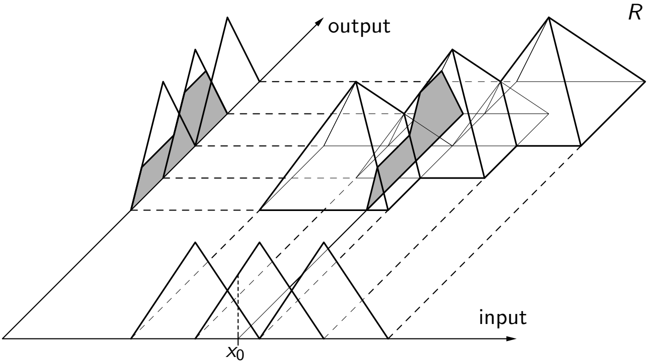
ns → ns', az → az', ps → ps'



$$R = \mu_{\text{ns}} \times \nu_{\text{ns'}} \cup \mu_{\text{az}} \times \nu_{\text{az'}} \cup \mu_{\text{ps}} \times \nu_{\text{ps'}}$$

Approximate Reasoning: Conclusion

Disjunctive Fuzzy Rules



3 fuzzy rules.

Every pyramid is specified by 1 fuzzy rule (Cartesian product).

Input x_0 leads to gray-shaded fuzzy output $\{x_0\} \circ R$.

Disjunctive or Conjunctive?

Fuzzy relation R employed in reasoning is obtained as follows.

For each rule i, we determine relation R_i by

$$R_i(x, y) = \min[M_i(x), N_i(y)]$$

for all $x \in X$, $y \in Y$.

Then, R is defined by union of R_i, *i.e.*

$$R = \bigcup_{1 \leq i \leq r} R_i.$$

That is, if-then rules are treated **disjunctive**.

If-then rules can be also treated **conjunctive** by

$$R = \bigcap_{1 \leq i \leq r} R_i.$$

Disjunctive or Conjunctive?

Decision depends on intended use and how R_i are obtained.

For both interpretations, two possible ways of applying composition:

$$B'_1 = A' \circ \left(\bigcup_{1 \le i \le r} R_i \right) \qquad B'_2 = A' \circ \left(\bigcap_{1 \le i \le r} R_i \right)$$

$$B'_3 = \bigcup_{1 \le i \le r} A' \circ R_i \qquad B'_4 = \bigcap_{1 \le i \le r} A' \circ R_i$$

Theorem

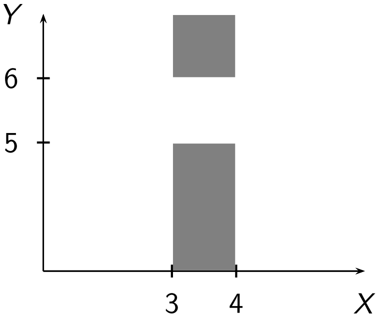
$B'_2 \subseteq B'_4 \subseteq B'_1 = B'_3$

This holds for any continuous $\top$ used in composition.

Approximate Reasoning
Conjunctive Imprecise Rules

if $X = [3, 4]$ **then** $Y = [5, 6]$

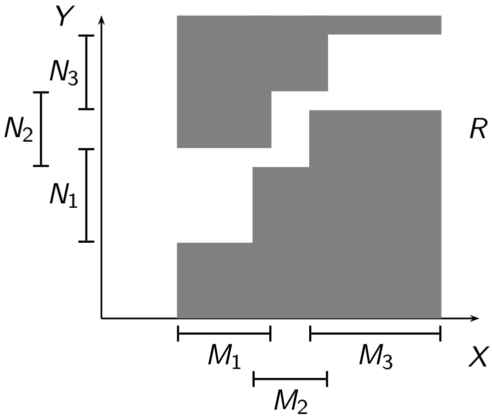
Gray-shaded values are impossible, white ones are possible.



Approximate Reasoning

Conjunctive Imprecise Rules

Several imprecise rules: **if** $X = M_1$ **then** $Y = N_1$, **if** $X = M_2$ **then** $Y = N_2$, **if** $X = M_3$ **then** $Y = N_3$.

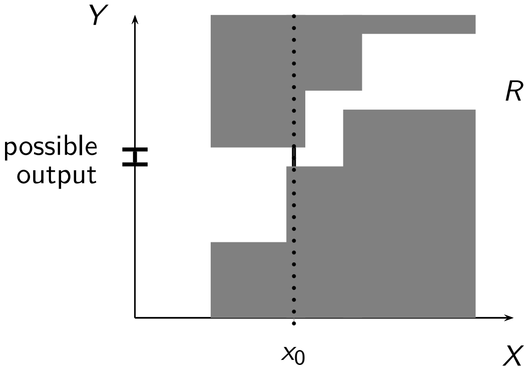


still possible are

$$R = \bigcap_{i=1}^{r} (M_i \times N_i) \cup (M_i^C \times Y)$$

"corridor" describing function's behavior

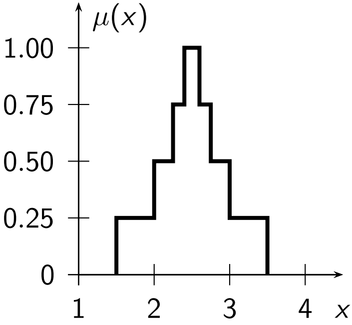
Approximate Reasoning with Crisp Input

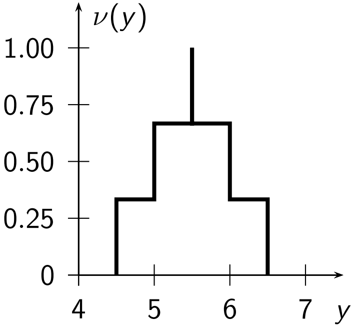


output $= \{x_0\} \circ R$

Generalization to Fuzzy Rules

if X is approx. 2.5 **then** Y is approx. 5.5

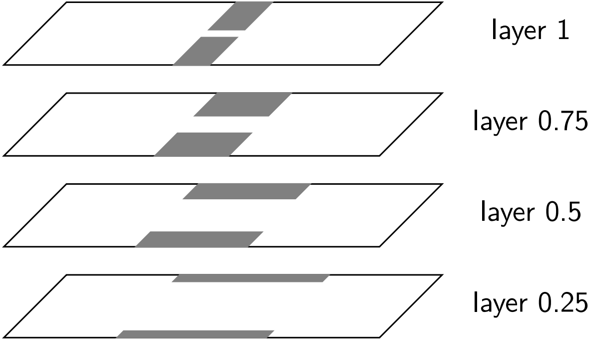




Modeling a Fuzzy Rule in Layers

Using Gödel Implication

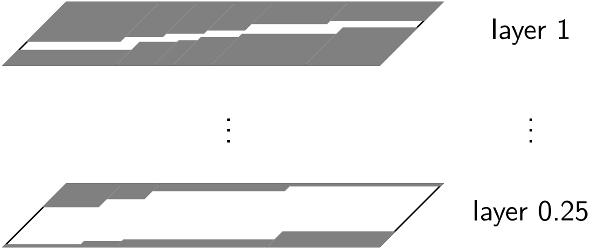
$$R_1 : \textbf{if } X = \mu_{M_1} \textbf{ then } Y = \nu_{B_1}$$



$$\mu_{R_1} : X \times Y \to [0,1], \quad I(x,y) = \begin{cases} 1 & \text{if } \mu_{M_1}(x) \leq \nu_{B_1}(y), \\ \nu_{B_1}(y) & \text{otherwise.} \end{cases}$$

Conjunctive Fuzzy Rule Base

R_1 : **if** $X = \mu_{M_1}$ **then** $Y = \nu_{B_1}$, $\quad \ldots, \quad$ R_n : **if** $X = \mu_{M_n}$ **then** $Y = \nu_{B_n}$



$$\mu_R = \min_{1 \le i \le r} \mu_{R_i}$$

Input μ_A, then output η with

$$\eta(y) = \sup_{x \in X} \min \{\mu_A(x), \mu_R(x, y)\}.$$

Example: Fuzzy Relation

Classes of cars $X = \{s, m, h\}$ (small, medium, high quality).

Possible maximum speeds $Y = \{140, 160, 180, 200, 220\}$ (in km/h).

Gor any $(x, y) \in X \times Y$, fuzzy relation ϱ states possibility that maximum speed of car of class x is y.

ϱ	140	160	180	200	220
s	1	.5	.1	0	0
m	0	.5	1	.5	0
h	0	0	.4	.8	1

Fuzzy Relational Equations

Given $\mu_1, \ldots, \mu_r$ of X and $\nu_1, \ldots, \nu_r$ of Y and r rules *if* μ_i *then* ν_i.

What is a **fuzzy relation** ϱ that fits the rule system?

One solution is to find a relation ϱ such that

$$\forall i \in \{1, \ldots, r\} : \nu_i = \mu_i \circ \varrho,$$

$$\mu \circ \varrho : Y \to [0, 1], \quad y \mapsto \sup_{x \in X} \min\{\mu(x),\ \varrho(x, y)\}.$$

Solution of a Relational Equation

Theorem

i) *Let "if A then B" be a rule with* $\mu_A \in \mathcal{F}(X)$ *and* $\nu_B \in \mathcal{F}(Y)$. *Then the relational equation* $\nu_B = \mu_A \circ \varrho$ *can be solved iff the Gödel relation* $\varrho_{A \bigcirc\!\!\!\sim B}$ *is a solution.*
$\varrho_{A \bigcirc\!\!\!\sim B} : X \times Y \to [0, 1]$ *is defined by*

$$(x, y) \mapsto \begin{cases} 1 & \text{if } \mu_A(x) \leq \nu_B(y), \\ \nu_B(y) & \text{otherwise.} \end{cases}$$

ii) *If* ϱ *is a solution, then the set of solutions* $R = \{\varrho_S \in \mathcal{F}(X \times Y) \mid \nu_B = \mu_A \circ \varrho_S\}$ *has the following property: If* $\varrho_{S'}, \varrho_{S''} \in R$, *then* $\varrho_{S'} \cup \varrho_{S''} \in R$.

iii) *If* $\varrho_{A \bigcirc\!\!\!\sim B}$ *is a solution, then* $\varrho_{A \bigcirc\!\!\!\sim B}$ *is the largest solution w.r.t.* $\subseteq$.

Example

$$\mu_A = \begin{pmatrix} .9 & 1 & .7 \end{pmatrix} \qquad \nu_B = \begin{pmatrix} 1 & .4 & .8 & .7 \end{pmatrix}$$

$$\varrho_{A \circledsim B} = \begin{pmatrix} 1 & .4 & .8 & .7 \\ 1 & .4 & .8 & .7 \\ 1 & .4 & 1 & 1 \end{pmatrix} \qquad \varrho_1 = \begin{pmatrix} 0 & 0 & 0 & .7 \\ 1 & .4 & .8 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{array}{ccc|cccc} & & & 1 & .4 & .8 & .7 \\ & & & 1 & .4 & .8 & .7 \\ & & & 1 & .4 & 1 & 1 \\ \hline .9 & 1 & .7 & 1 & .4 & .8 & .7 \end{array} \qquad \varrho_2 = \begin{pmatrix} 0 & .4 & .8 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & .7 \end{pmatrix}$$

$\varrho_{A \circledsim B}$ largest solution, ϱ_1, ϱ_2 are two minimal solutions.

Solution space forms upper semilattice.

Solution of a Set of Relational Equations

Generalization of this result to system of r relational equations:

Theorem

Let $\nu_{B_i} = \mu_{A_i} \circ \varrho$ for $i = 1, \ldots, r$ be a system of relational equations.

i) There is a solution iff $\bigcap_{i=1}^{r} \varrho_{A_i \circledast B_i}$ is a solution.

ii) If $\bigcap_{i=1}^{r} \varrho_{A_i \circledast B_i}$ is a solution, then this solution is the biggest solution w.r.t. $\subseteq$.

Remark: if there is no solution, then Gödel relation is often at least a good approximation.

Solving a System of Relational Equations

Sometimes it is a good choice not to use the largest but a smaller solution.
i.e. the **Cartesian product** $\varrho_{A\times B}(x, y) = \min\{\mu_A(x), \nu_B(y)\}$.

If a solution of the relational equation $\nu_B = \mu_A \circ \varrho$ for ϱ exists, then $\varrho_{A\times B}$ is a solution, too.

Theorem

Let $\mu_A \in \mathcal{F}(X)$, $\nu_B \in \mathcal{F}(Y)$. *Furthermore, let* $\varrho \in \mathcal{F}(X \times Y)$ *be a fuzzy relation which satisfies the relational equation* $\nu_B = \mu_A \circ \varrho$. *Then* $\nu_B = \mu_A \circ \varrho_{A\times B}$ *holds.*

Solving a System of Relational Equations

Using Cartesian product

$\mu_{A_i} = \nu_{B_i} \circ \varrho,\ 1 \leq i \leq r$ can be reasonably solved with $A \times B$ by

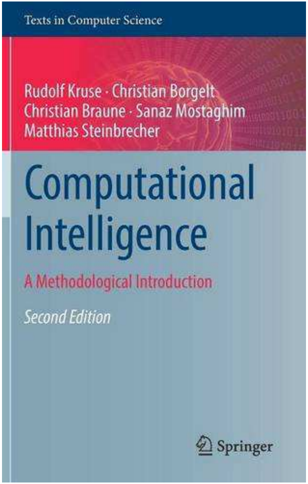
$$\varrho = \max\left\{\varrho_{A_i \times B_i} \mid 1 \leq i \leq r\right\}.$$

For crisp value $x_0 \in X$ (represented by $\mathbb{1}_{\{x_0\}}$):

$$\begin{aligned}
\nu(y) &= \left(\mathbb{1}_{\{x_0\}} \circ \varrho\right)(y) \\
&= \max_{1\leq i\leq r}\left\{\sup_{x\in X} \min\{\mathbb{1}_{\{x_0\}}(x), \varrho_{A_i\times B_i}(x,y)\}\right\} \\
&= \max_{1\leq i\leq r}\{\min\{\mu_{A_i}(x_0), \nu_{B_i}(y)\}.\}
\end{aligned}$$

That is Mamdani-Assilian fuzzy control (to be discussed).

References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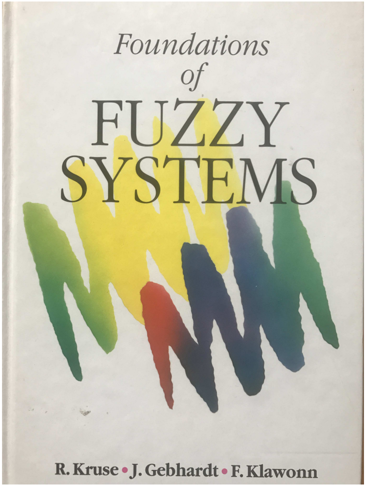
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R. Kruse • J. Gebhardt • F. Klawonn