

### Exercise Sheet 5

#### Semi-Graphoid and Graphoid Axioms

Unusually, one requires any notion of conditional independence to satisfy as a minimum the so-called *Semi-Graphoid axioms*: Let  $W$ ,  $X$ ,  $Y$  and  $Z$  be disjoint sets of attributes, with  $W$ ,  $X$  and  $Y$  being non-empty.  $X \perp\!\!\!\perp Y \mid Z$  shall denote „ $X$  is conditionally independent of  $Y$  given  $Z$ .“

Symmetry	$X \perp\!\!\!\perp Y \mid Z \implies Y \perp\!\!\!\perp X \mid Z$
Decomposition	$W \cup X \perp\!\!\!\perp Y \mid Z \implies X \perp\!\!\!\perp Y \mid Z$
Weak Union	$W \cup X \perp\!\!\!\perp Y \mid Z \implies X \perp\!\!\!\perp Y \mid Z \cup W$
Contraction	$(W \perp\!\!\!\perp X \mid Z) \wedge (W \perp\!\!\!\perp Y \mid Z \cup X)$ $\implies W \perp\!\!\!\perp X \cup Y \mid Z$

It is pleasant to also have the following axiom satisfied:

Intersection	$(W \perp\!\!\!\perp X \mid Z \cup Y) \wedge (W \perp\!\!\!\perp Y \mid Z \cup X)$ $\implies W \perp\!\!\!\perp X \cup Y \mid Z$
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All five axioms together are referred to as the *Graphoid axioms*. One can show that the conditional stochastic independence for strictly positive probability distributions satisfies the Graphoid axioms.

#### Exercise 16      Semi-Graphoid and Graphoid Axioms

Show that the conditional stochastic (probabilistic) independence satisfies the decomposition axiom!

(Hint: In the probabilistic case  $X \perp\!\!\!\perp Y \mid Z$  means that

$$\forall x, y, z : \quad P(X = x, Y = y \mid Z = z) = P(X = x \mid Z = z) \cdot P(Y = y \mid Z = z)$$

or, equivalently, that

$$\forall x, y, z : \quad P(X = x \mid Y = y, Z = z) = P(X = x \mid Z = z).$$

The proof can be accomplished by inserting these relations and applying the well-known Kolmogorov axioms)

#### Exercise 17      Semi-Graphoid and Graphoid Axioms

Show that the conditional stochastic (probabilistic) independence satisfies the weak union axiom!

#### Exercise 18      Semi-Graphoid and Graphoid Axioms

Show that the conditional stochastic (probabilistic) independence does **not** satisfy the intersection axiom if we allow 0 probabilities!