Fuzzy Set - Basics

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Motivation

Every day humans use imprecise linguistic terms
e.g. big, *fast*, *about 12 o'clock*, *old*, etc.

All complex human actions are decisions based on such concepts:

- driving and parking a car,
- financial/business decisions,
- law and justice,
- ….

So, these terms and the way they are processed play a crucial role for the development of intelligent systems .

Computers need a mathematical model to express and process such complex semantics.

Concepts of classical mathematics are often inadequate for such models.

Example – The Sorites Paradox

If a pile of sand is small, adding one grain of sand to it leaves it small.
A pile of sand with a single grain is small.

Hence all sand dunes are small.

Paradox comes from an all-or-nothing treatment of *small*.

It is better to say that the degree of truth of the statement "pile of sand is small" decreases by adding one grain after another.

Example - Solution of the Sorites Paradox

Statement $A(n)$: “n grains of sand form a sand pile.”

Let $d_n = T(A(n))$ denote “degree of truth” for $A(n)$. Then

$$0 = d_0 \leq d_1 \leq \ldots \leq d_n \leq \ldots \leq 1$$

can be seen as truth values of a **many valued logic**

with 0 (false) and 1 (true).

The concept of a Fuzzy Logic is useful in such cases

(not everything is either black or white, often there are degrees or grey-levels)

Example - Imprecise Linguistic Expressions

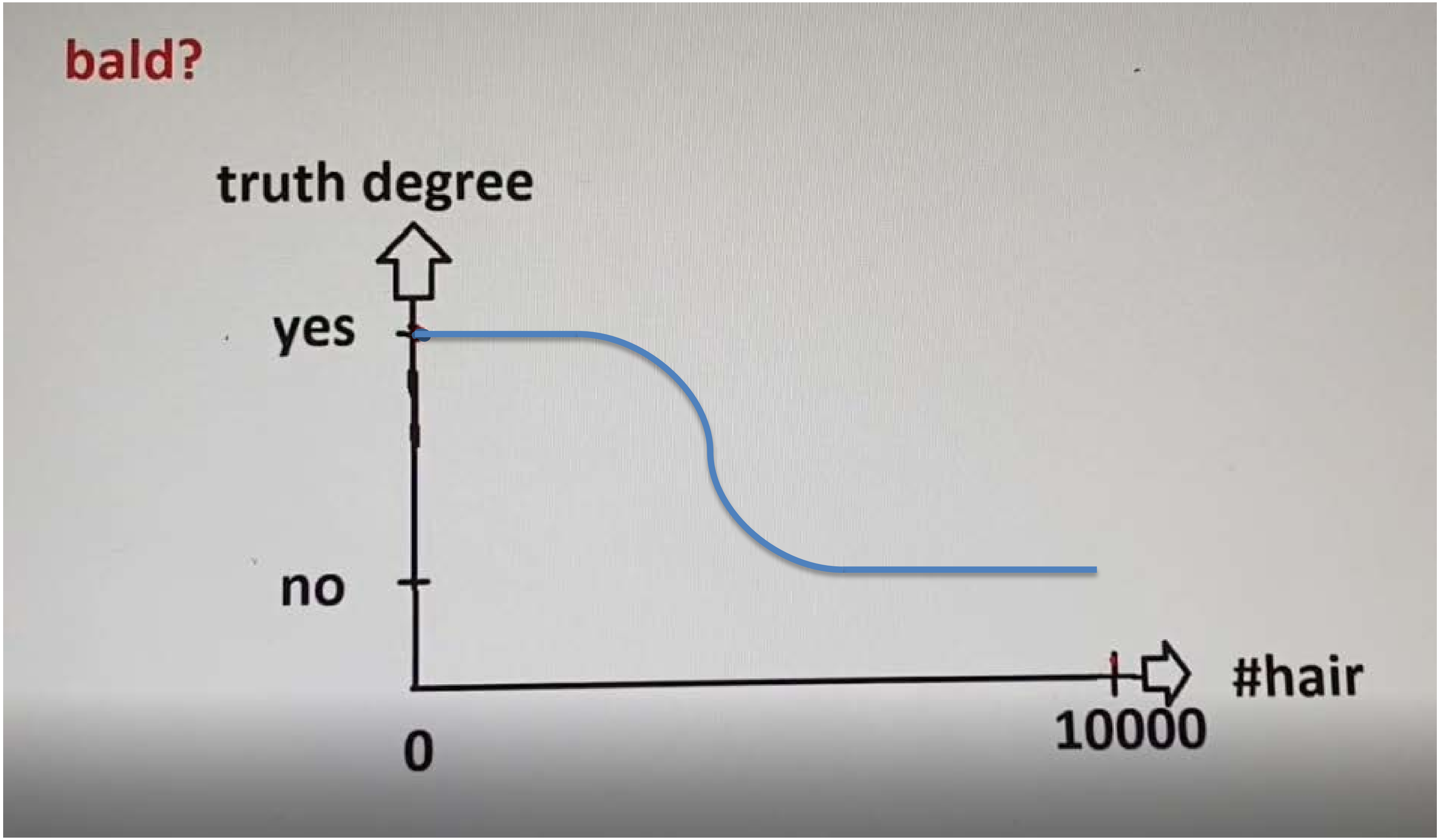
Consider the notion *bald*:

A person without hair on his head is bald,
a very hairy person is definitely not bald.

Usually, *bald* is only partly applicable to a person.

A *baldness/non baldness* threshold is counter-intuitive?

Solution: Use „generalized sets“ with membership degrees.



The concept of a Fuzzy set is useful in such cases.

Lotfi Asker Zadeh

Classes of objects in the real world do not have precisely defined criteria of membership.

Such imprecisely defined “classes” play an important role in human thinking and the development of intelligent systems.

Particularly in domains of pattern recognition, communication of information, and abstraction.

Lotfi A. Zadeh's Principle of Incompatibility

"Stated informally, the essence of this principle is that as the complexity of a system increases, our ability to make precise and yet significant statements about its behavior diminishes until a threshold is reached beyond which precision and significance (or relevance) become almost mutually exclusive characteristics."

Fuzzy sets/fuzzy logic are used as mechanism for abstraction of unnecessary or too complex details.

Applications using Fuzzy Sets

Control Engineering

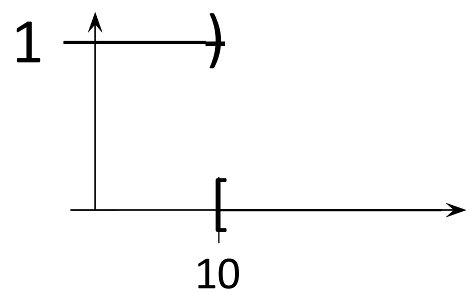
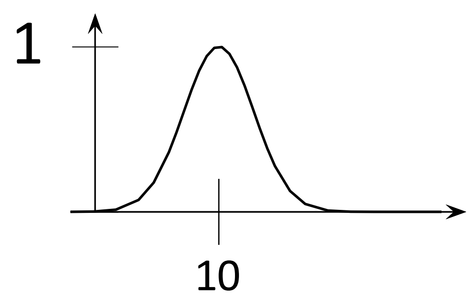
Approximate Reasoning

Data Sciences

Artificial Intelligence

Rudolf Kruse received IEEE Fuzzy Pioneer Award for „Learning Methods for Fuzzy Systems“ in 2018

Fuzzy sets as generalizations of classical sets

ling. description		model	
all numbers smaller than 10	*objective*		characteristic function of a set
all numbers <u>almost</u> equal to 10	*subjective*		membership function of a “fuzzy set”

Definition

A fuzzy set μ of *X* is a function from the reference set *X* to the unit interval, *i.e.* $\mu : X \to [0, 1]$. $F(X)$ represents the set of all fuzzy sets of *X*, *i.e.* $F(X) := \{\mu \mid \mu : X \to [0, 1]\}$.

Membership Functions

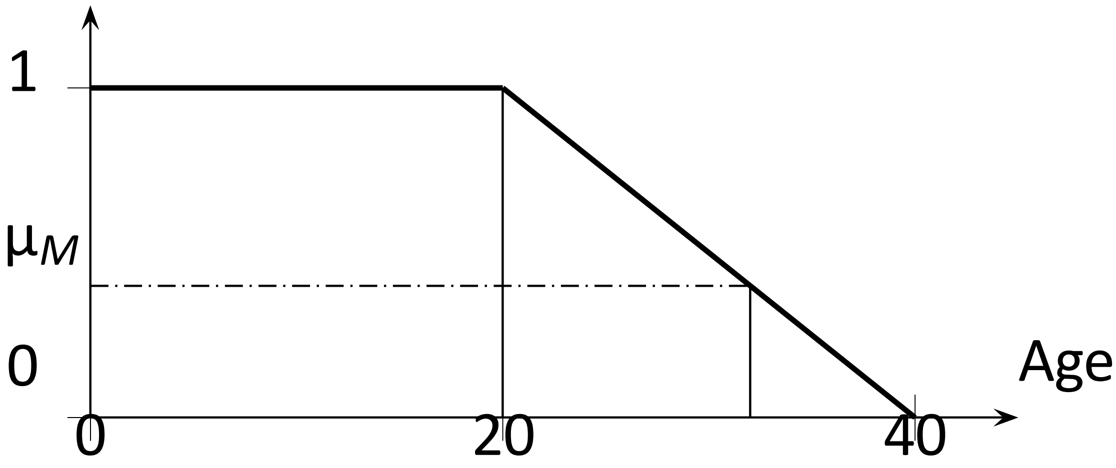
$\mu_M(u) = 1$ reflects full membership in M.

$\mu_M(u) = 0$ expresses absolute non-membership in M.

Sets can be viewed as special case of fuzzy sets where only full membership and absolute non-membership are allowed.

Such sets are called *crisp sets* or Boolean sets.

Membership degrees $0 < \mu_M < 1$ represent *partial membership*.



Representing *young* in “a young person”

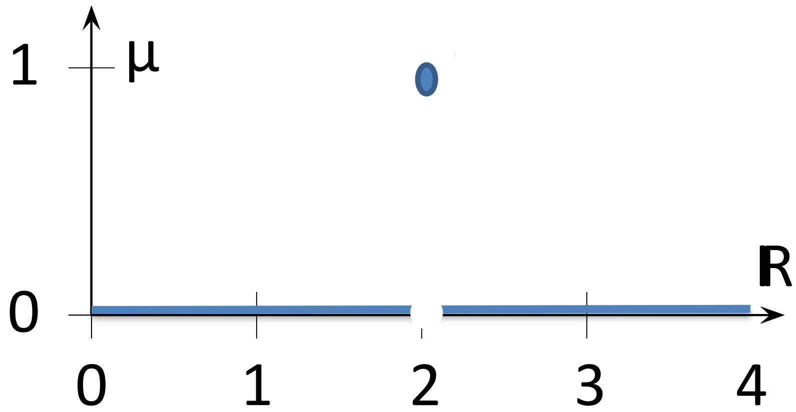
Membership Functions

A membership function attached to a given linguistic description (such as *young*) depends on the context – it is subjective.

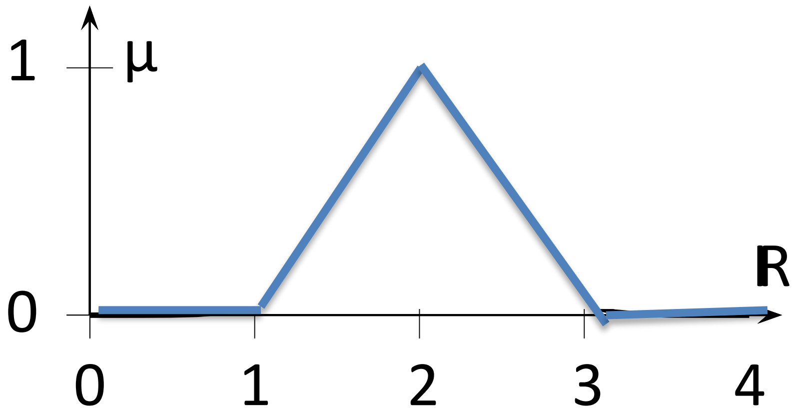
A young retired person is certainly older than a young student.
Even the idea of young student depends on the user.

Membership degrees are fixed only *by convention*:
Unit interval as range of membership grades is arbitrary but easy to use.

Examples: Fuzzy Sets

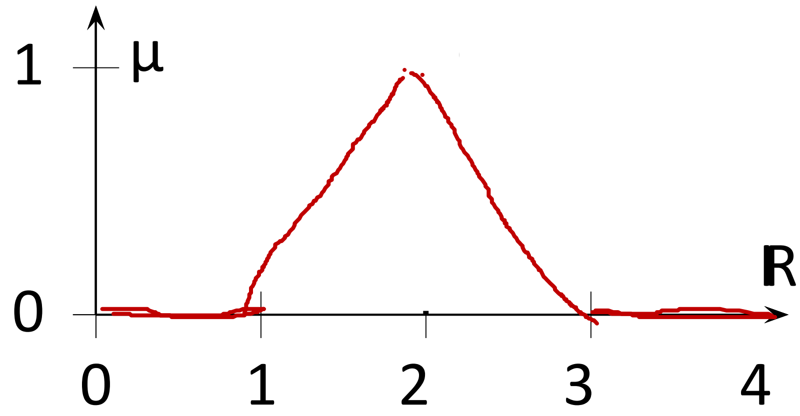


exactly two

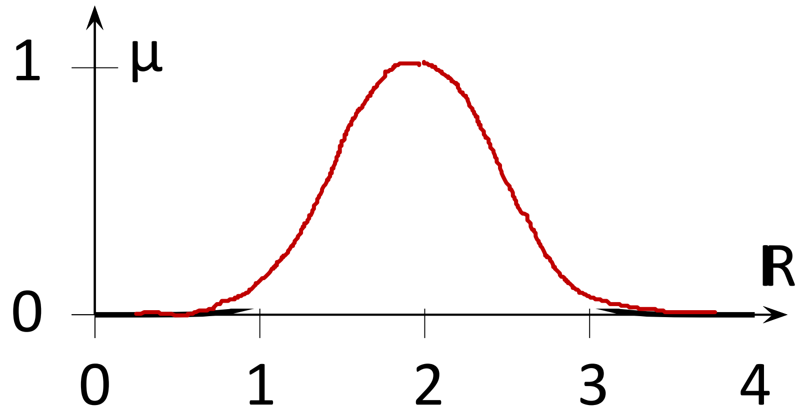


between 1 and 3

Examples: Fuzzy Sets



Approximately 2



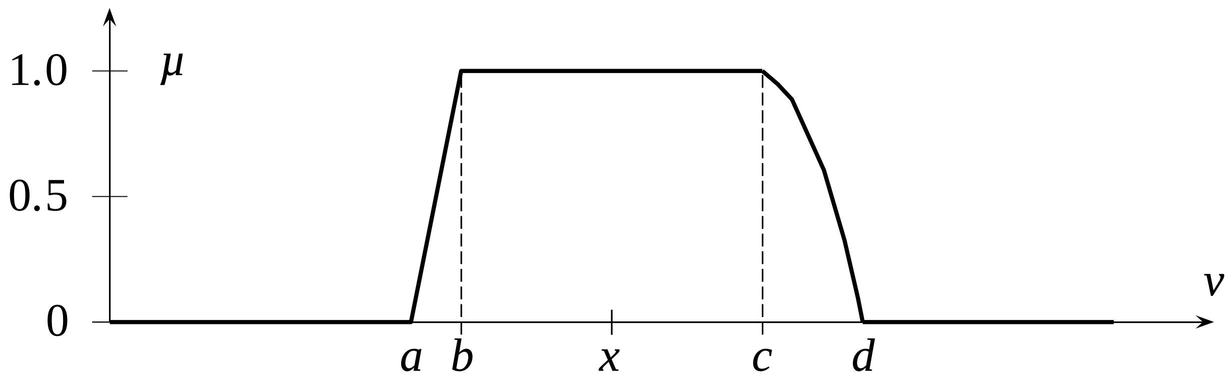
Approximately 2

Exact numerical value has membership degree of 1.

Left: monotonically increasing, right: monotonically decreasing, *i.e.* unimodal function.

Terms like *around* modeled using triangular or Gaussian function.

Example – Velocity of Rotating Hard Disk



Fuzzy set μ characterizing the normal velocity of rotating hard disk.

Let *v* be the velocity of rotating hard disk in revolutions per minute.

Modelling of expert's knowledge:

"It's *impossible* that *v* drops under *a* or exceeds *d* .

It's highly certain that any value between [*b*, *c*] can occur."

Otherwise I defined my subjective point of view , I also used data"

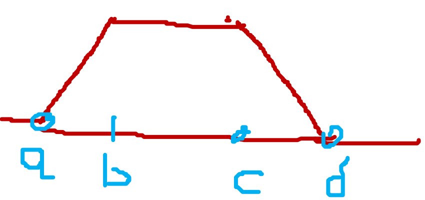
Vertical Representation

So far, fuzzy sets were described by
their characteristic/membership function and
assigning degree of membership $\mu(x)$ to each element $x \in X$.

That is the **vertical representation** of the corresponding fuzzy set,
e.g. linguistic expression like "about m"

$$\mu_{m,d}(x) = \begin{cases} 1 - \left|\frac{m-x}{d}\right|, & \text{if } m - d \leq x \leq m + d \\ 0, & \text{otherwise,} \end{cases}$$

or "approximately between b and c"



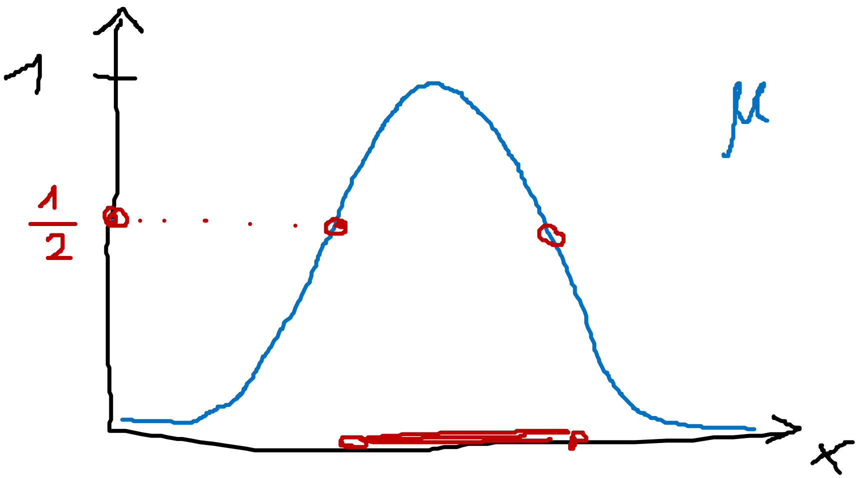
$$\mu_{a,b,c,d}(x) = \begin{cases} \frac{x-a}{b-a}, & \text{if } a \leq x < b \\ 1, & \text{if } b \leq x \leq c \\ \frac{x-d}{c-d}, & \text{if } c < x \leq d \\ 0, & \text{if } x < a \text{ or } x > d. \end{cases}$$

Level Sets (cuts) for a Fuzzy set

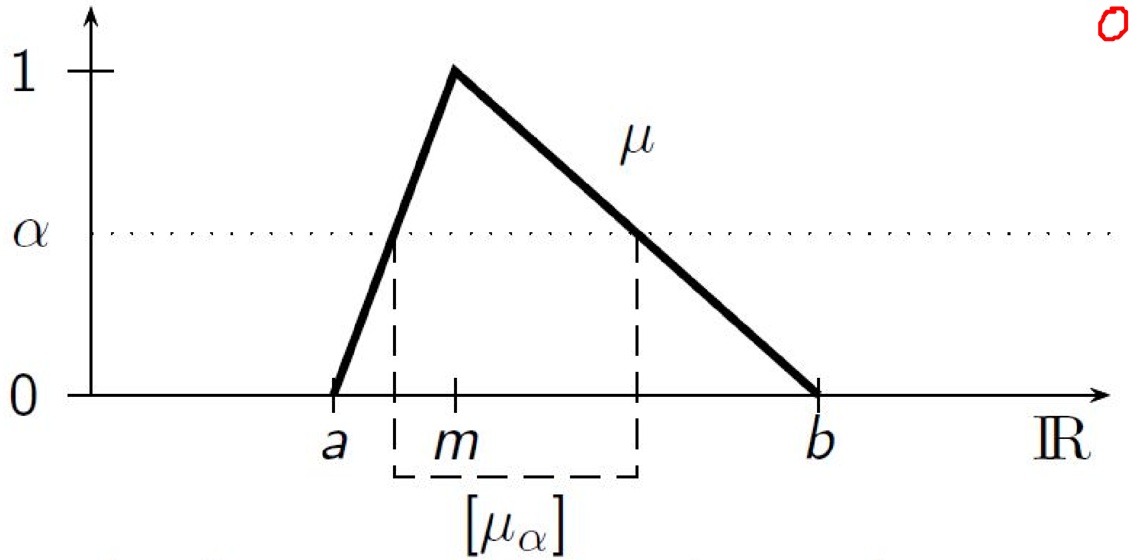
Let $\mu \in \mathcal{F}(X)$ and $\alpha \in [0, 1]$. Then the sets

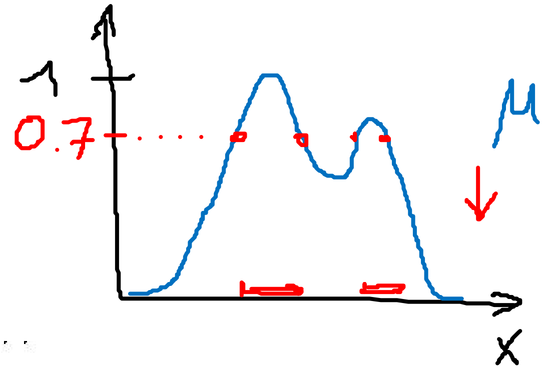
$$[\mu]_\alpha = \{x \in X \mid \mu(x) \geq \alpha\}, \quad [\mu]_{\underline{\alpha}} = \{x \in X \mid \mu(x) > \alpha\}$$

are called the α-*cut* and *strict* α-*cut* of μ.



An Example





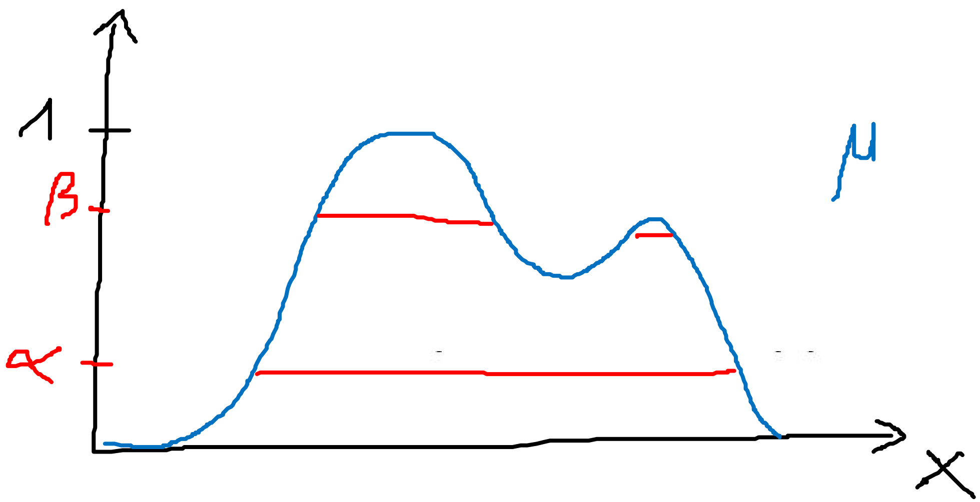
Let μ be triangular function on $\mathbb{R}$ as shown above.

α-cut of μ can be constructed by

1. drawing horizontal line parallel to x-axis through point $(0, \alpha)$,
2. projecting this section onto x-axis.

$$[\mu]_\alpha = \begin{cases} [a + \alpha(m - a), b - \alpha(b - m)], & \text{if } 0 < \alpha \leq 1, \\ \mathbb{R}, & \text{if } \alpha = 0. \end{cases}$$

Properties of α-cuts I



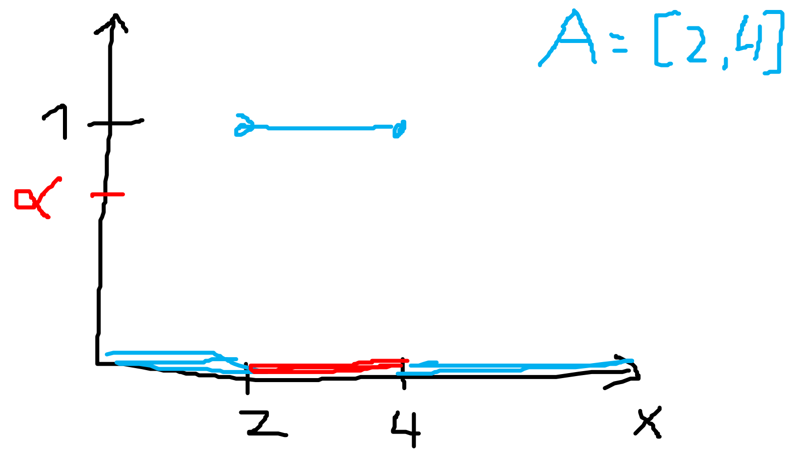
Theorem
Let $\mu \in \mathcal{F}(X)$, $\alpha \in [0,1]$ *and* $\beta \in [0,1]$.

(a) $[\mu]_0 = X$,

(b) $\alpha < \beta \Longrightarrow [\mu]_\alpha \supseteq [\mu]_\beta$,

(c) $\bigcap_{\alpha:\alpha<\beta} [\mu]_\alpha = [\mu]_\beta$.

Characteristic function



Let $A \subseteq X$, $\chi_A : X \to [0, 1]$

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{otherwise} \end{cases}$$

Then $[\chi_A]_\alpha = A$ for $0 < \alpha \leq 1$.

χ_A is called indicator function or characteristic function of A.

Properties of α-cuts II

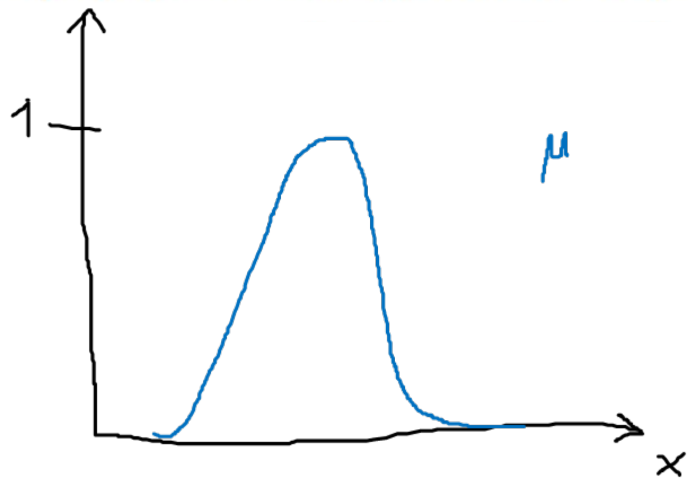
Theorem (Representation Theorem)
Let $\mu \in \mathcal{F}(X)$. Then

$$\mu(x) = \sup_{\alpha \in [0,1]} \left\{ \min(\alpha, \chi_{[\mu]_\alpha}(x)) \right\}$$

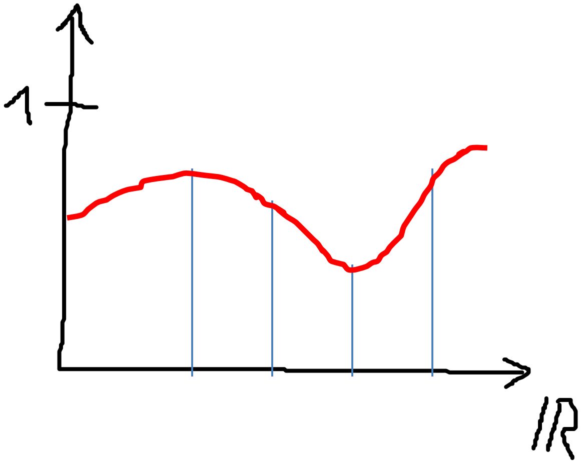
where $\chi_{[\mu]_\alpha}(x) = \begin{cases} 1, & \text{if } x \in [\mu]_\alpha \\ 0, & \text{otherwise.} \end{cases}$

So, fuzzy set can be obtained as upper envelope of its α-cuts.

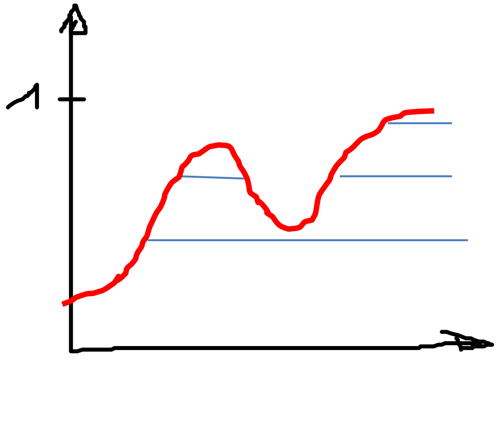
Simply draw α-cuts parallel to horizontal axis in height of α.



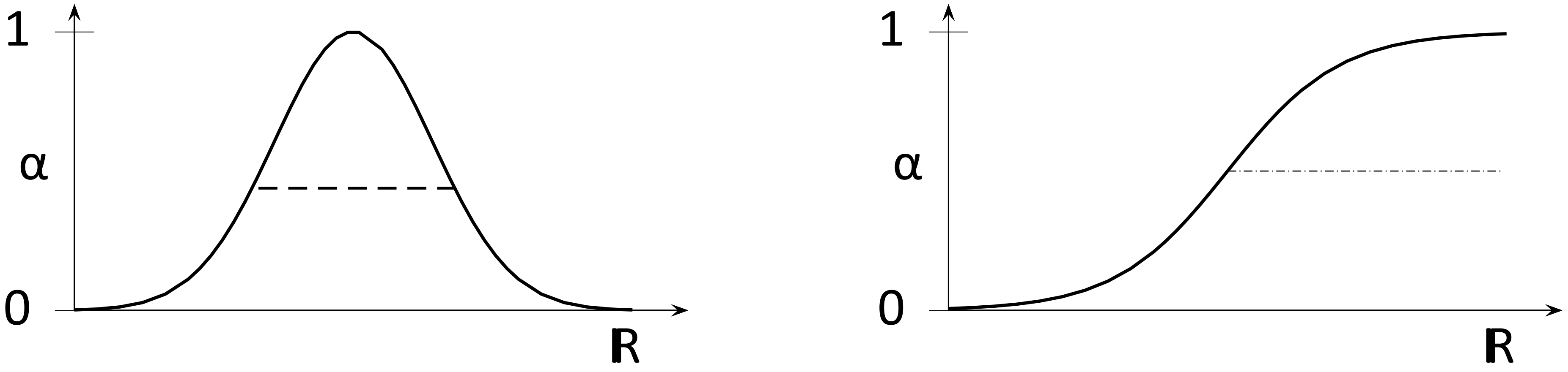
Vertical View



Horizontal View



Convex Fuzzy Sets



A fuzzy set μ ∈ F(IR) *is convex if and only if*

$$\mu(\lambda x_1 + (1 - \lambda)x_2) \geq \min\{\mu(x_1), \mu(x_2)\}$$

for all $x_1, x_2 \in$ IR *and all* $\lambda \in [0, 1]$.

System of Sets

In this manner we obtain **system of sets**

$$\mathcal{A} = (A_\alpha)_{\alpha \in L}, \quad L \subseteq [0,1], \quad \text{card}(L) \in \mathbb{N}.$$

$\mathcal{A}$ must satisfy consistency conditions for $\alpha, \beta \in L$:

(a) $0 \in L \Longrightarrow A_0 = X,$ (fixing of reference set)

(b) $\alpha < \beta \Longrightarrow A_\alpha \supseteq A_\beta.$ (monotonicity)

This induces fuzzy set

$$\mu_{\mathcal{A}} : X \to [0,1],$$
$$\mu_{\mathcal{A}}(x) = \sup_{\alpha \in L} \left\{ \min(\alpha, \chi_{A_\alpha}(x)) \right\}.$$

If L is not finite but comprises all values $[0,1]$, then μ must satisfy

(c) $\bigcap_{\alpha : \alpha < \beta} A_\alpha = A_\beta.$ (condition for continuity)

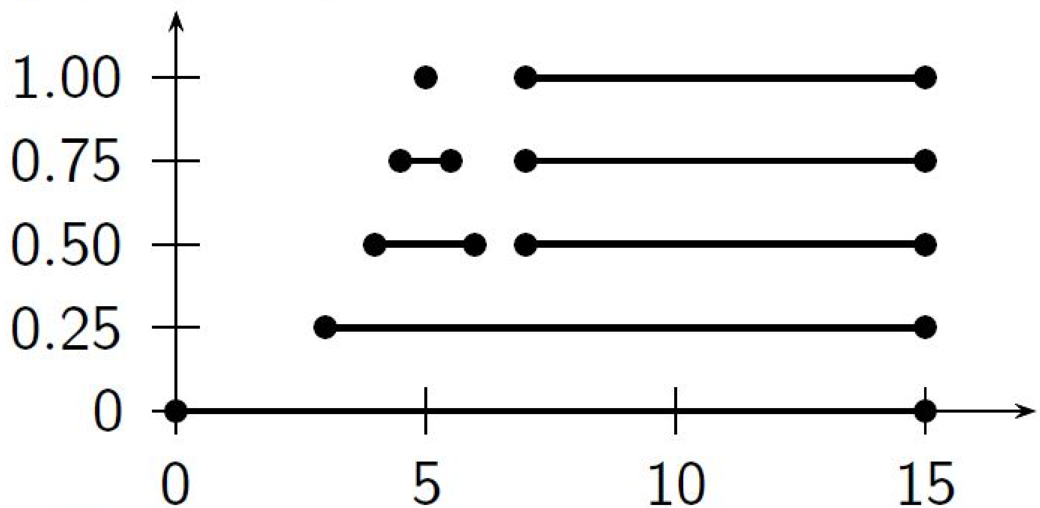
“Approximately 5 or greater than or equal to 7”

An Exemplary Horizontal View

Suppose that $X = [0, 15]$.

An expert chooses $L = \{0, 0.25, 0.5, 0.75, 1\}$ and α-cuts:

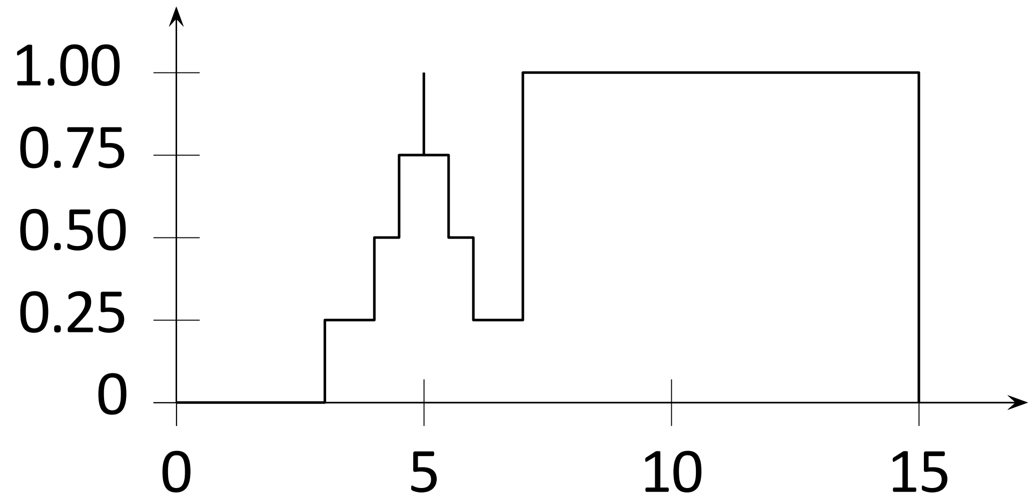
- $A_0 \quad = [0, 15]$,
- $A_{0.25} = [3, 15]$,
- $A_{0.5} \; = [4, 6] \cup [7, 15]$,
- $A_{0.75} = [4.5, 5.5] \cup [7, 15]$,
- $A_1 \quad = \{5\} \cup [7, 15]$.



The family $(A_\alpha)_{\alpha \in L}$ of sets induces upper shown fuzzy set.

μ_A is obtained as upper envelope of the family A of sets.

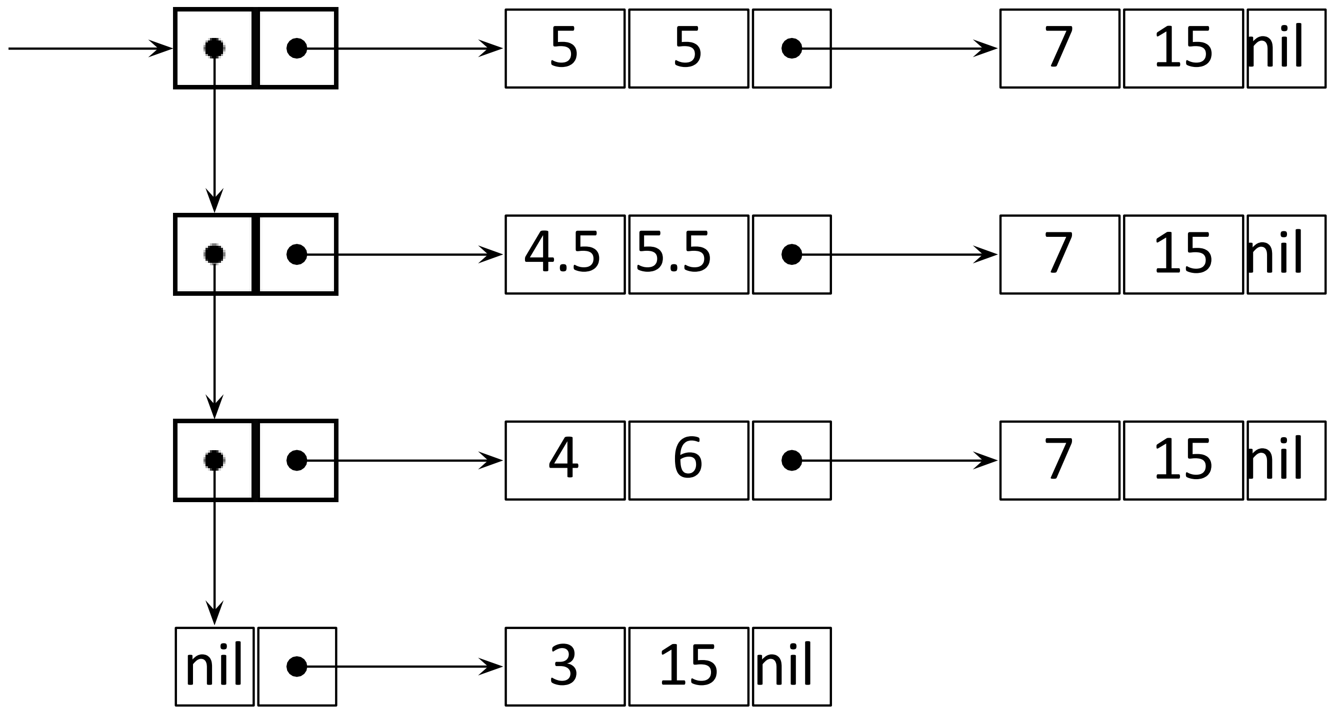
The difference between horizontal and vertical view is obvious:



The horizontal representation is easier to process in computers.

Also, restricting the domain of x-axis to a discrete set is usually done.

Horizontal Representation in the Computer



Fuzzy sets are usually stored as chain of linear lists.

A finite union of closed intervals is stored by their bounds.

This data structure is appropriate for arithmetic operators.