Fuzzy Relations

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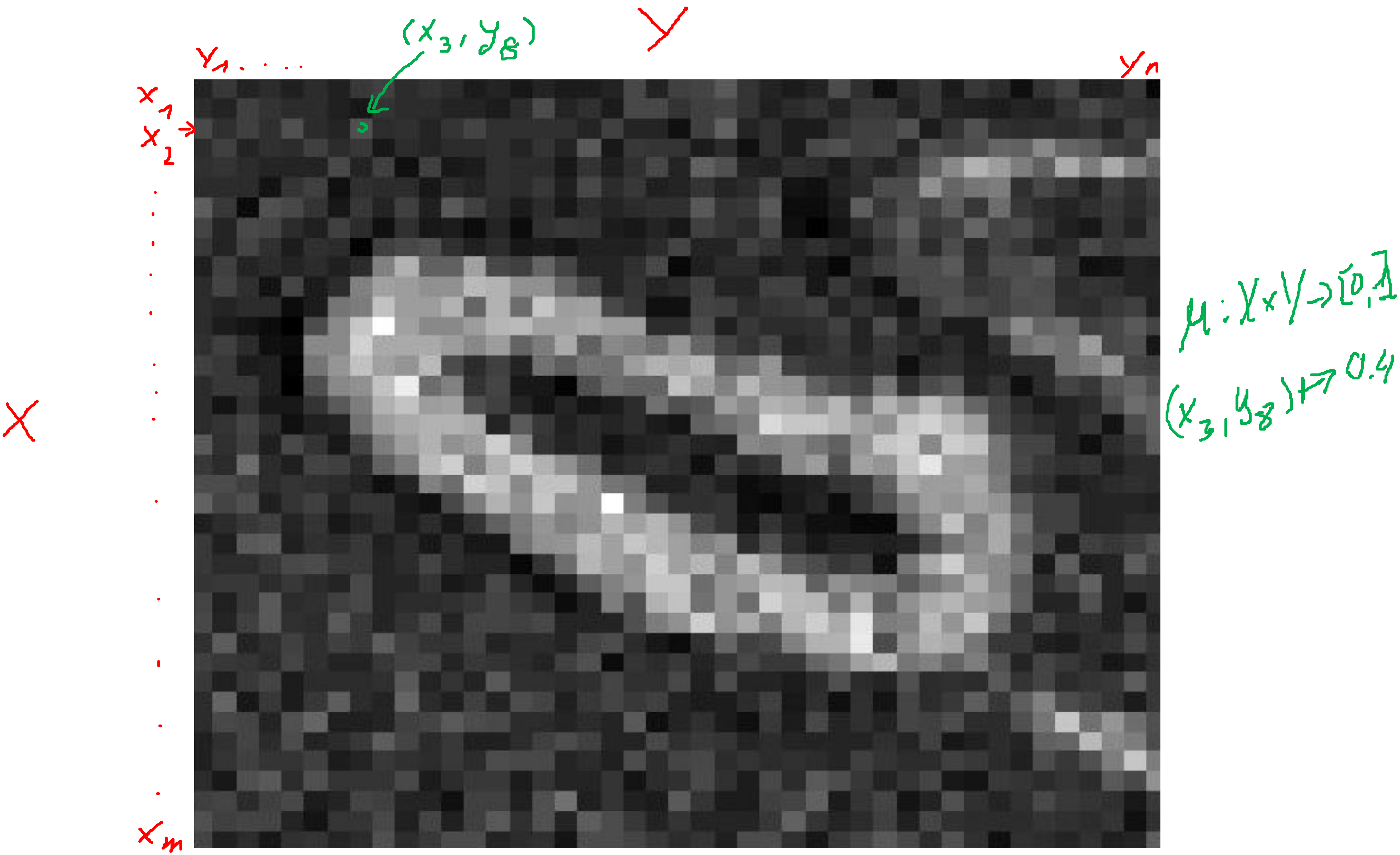
A grey level picture interpreted as a fuzzy set



Definition of Relation

A **relation** among crisp sets $X_1, \ldots, X_n$ is a subset of the Cartesian Product $X_1 \times \ldots \times X_n$. It is denoted as $R(X_1, \ldots, X_n)$ or $R(X_i \mid 1 \leq i \leq n)$.

So, the relation $R(X_1, \ldots, X_n) \subseteq X_1 \times \ldots \times X_n$ is set, too. The

basic concept of sets can be also applied to relations:

- containment, subset, union, intersection, complement

Each crisp relation can be defined by its characteristic function

$$R(x_1, \ldots, x_n) = \begin{cases} 1, & \text{if and only if } (x_1, \ldots, x_n) \in R, \\ 0, & \text{otherwise.} \end{cases}$$

The membership of $(x_1, \ldots, x_n)$ in R indicates whether the elements of $(x_1, \ldots, x_n)$ are related to each other or not.

Fuzzy Relations

The characteristic function of a crisp relation can be generalized to allow tuples to have degrees of membership.

A **fuzzy relation R** is a fuzzy set of $X_1 \times . . . \times X_n$

The membership grade indicates strength of the present relation between elements of the tuple.

The fuzzy relation can also be represented by an *n*-dimensional membership array.

Example

Let R be a fuzzy relation between two sets X = {New York City, Paris} and Y = {Beijing, New York City, London}.

R shall represent relational concept "very far".

It can be represented (subjectively) as two-dimensional membership array:

	NYC	Paris
Beijing	1	0.9
NYC	0	0.7
London	0.6	0.3

Cartesian Product of Fuzzy Sets: *n* Dimensions

Let $A_1, \dots, A_n$ be fuzzy sets (n ≥ 2) in $X_1, \dots, X_n$, respectively

The (fuzzy) *Cartesian product* of $A_1, \dots, A_n,$ denoted by $A_1 \times \dots \times A_n,$ is a fuzzy relation of the product space $X_1 \times \dots \times X_n$.

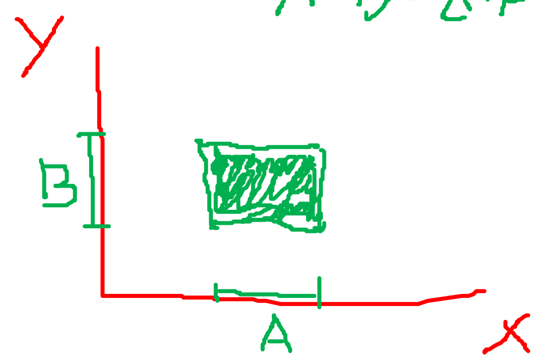
It is defined by its membership function

$$\mu_{A_1 \times \dots \times A_n}(x_1, \dots, x_n) = T\,(\mu_{A_1}(x_1), \dots, \mu_{A_n}(x_n))$$

$$\textit{for } x_i \in X_i,\ 1 \le i \le n.$$

In most applications T = min is chosen.

$A \times B = \{(a,b) \mid a \in A, b \in B\}$



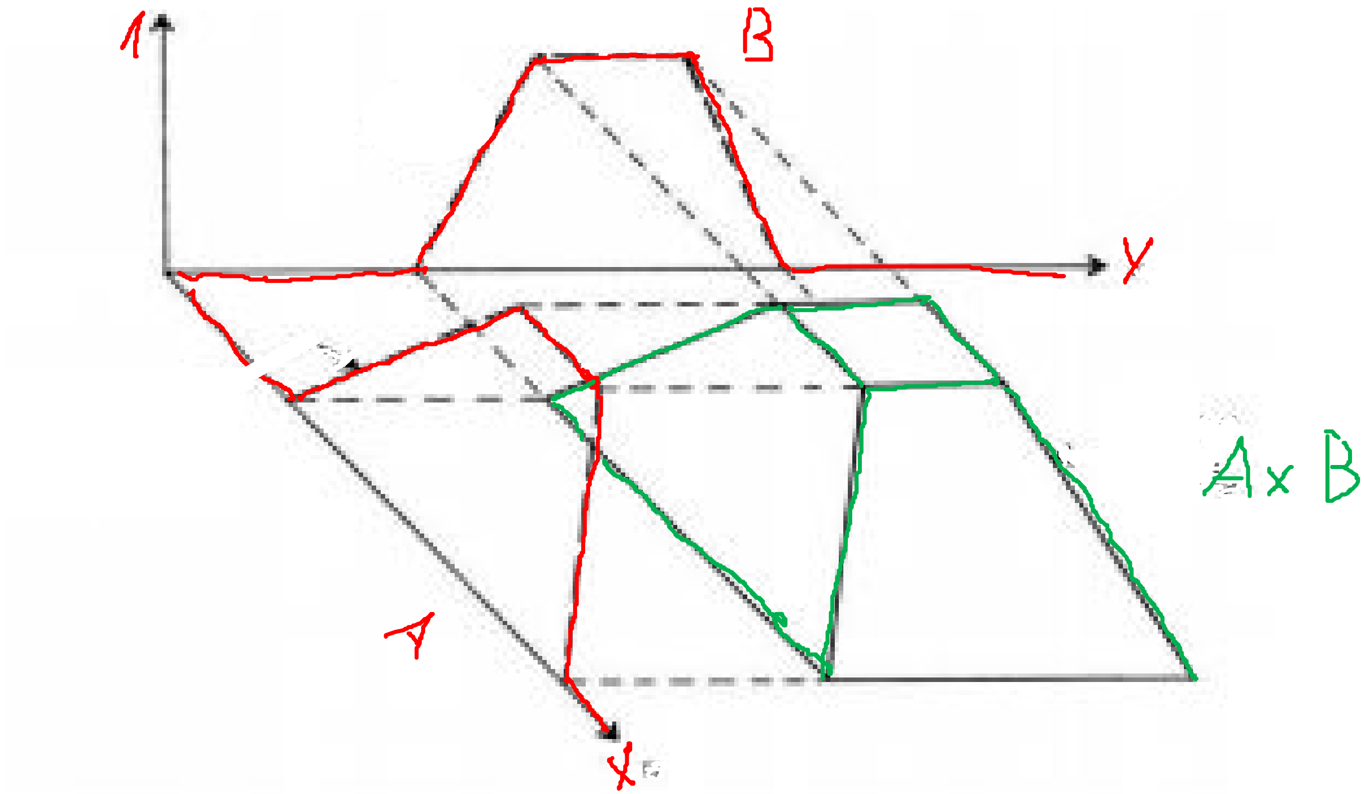
Cartesian Product of Fuzzy Sets in two Dimensions

A special case of the Cartesian product is when $n = 2$.

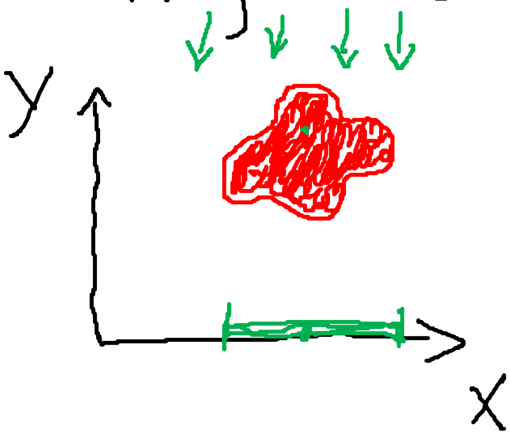
Then the Cartesian product of fuzzy sets $A \in \mathrm{F}(X)$ and $B \in \mathrm{F}(Y)$ is a fuzzy relation $A \times B \in \mathrm{F}(X \times Y)$ defined by

$$\mu_{A \times B}(x, y) = \top\,[\mu_A(x),\ \mu_B(y)],\ \text{for all } x \in X \text{, and } y \in Y.$$

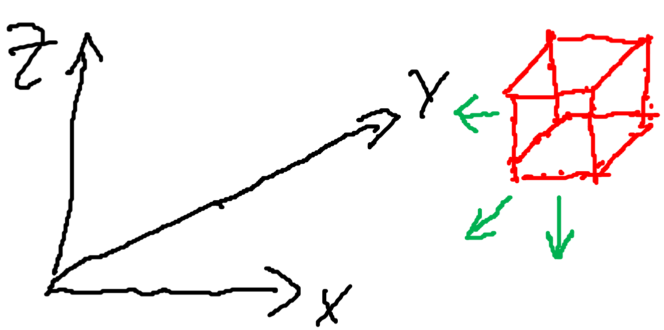
Example: Cartesian Product in $F(X \times Y)$ with t-norm = min



Projection

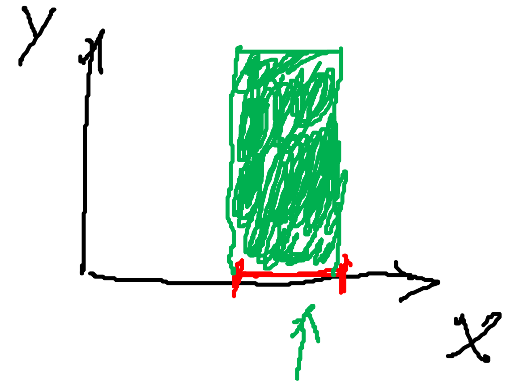


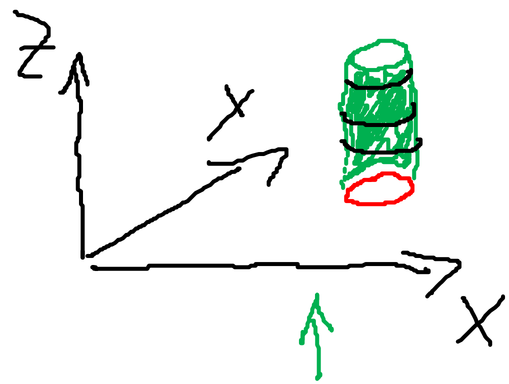
2 projections



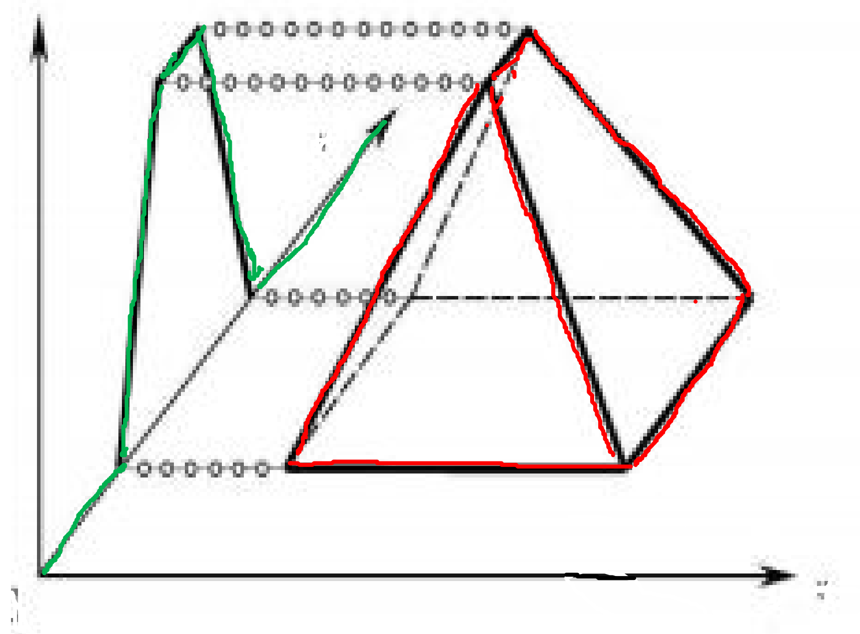
6 projections

Cylindrical Extension

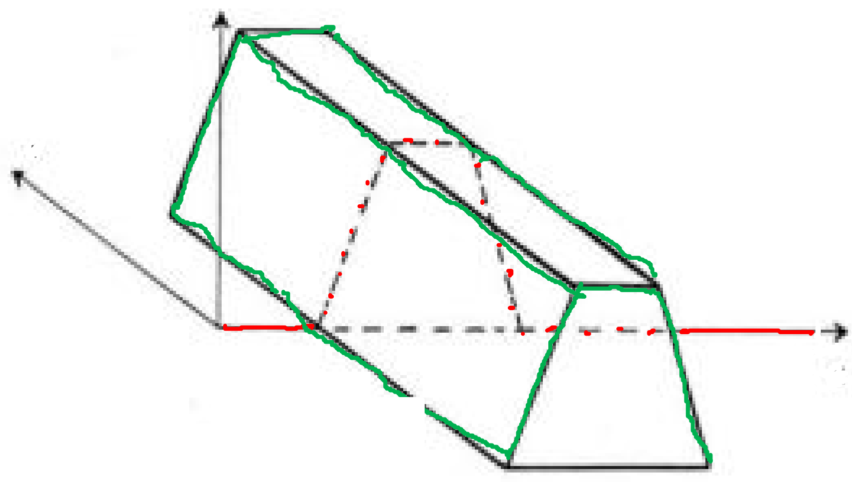




projection of μ



cylindrical extension of μ



Example

Consider the sets $X_1 = \{0, 1\}$, $X_2 = \{0, 1\}$, $X_3 = \{0, 1, 2\}$
and the ternary fuzzy relation on $X_1 \times X_2 \times X_3$:

Let $R_{ij} = [R \downarrow \{X_i, X_j\}]$ and $R_i = [R \downarrow \{X_i\}]$ for all $i, j \in \{1, 2, 3\}$.

Using this notation, all possible projections of R are given below.

$(x_1,$	$x_2,$	$x_3)$	$R(x_1, x_2, x_3)$	$R_{12}(x_1, x_2)$	$R_{13}(x_1, x_3)$	$R_{23}(x_2, x_3)$	$R_1(x_1)$	$R_2(x_2)$	$R_3(x_3)$
0	0	0	0.4	0.9	1.0	0.5	1.0	0.9	1.0
0	0	1	0.9	0.9	0.9	0.9	1.0	0.9	0.9
0	0	2	0.2	0.9	0.8	0.2	1.0	0.9	1.0
0	1	0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
0	1	1	0.0	1.0	0.9	0.5	1.0	1.0	0.9
0	1	2	0.8	1.0	0.8	1.0	1.0	1.0	1.0
1	0	0	0.5	0.5	0.5	0.5	1.0	0.9	1.0
1	0	1	0.3	0.5	0.5	0.9	1.0	0.9	0.9
1	0	2	0.1	0.5	1.0	0.2	1.0	0.9	1.0
1	1	0	0.0	1.0	0.5	1.0	1.0	1.0	1.0
1	1	1	0.5	1.0	0.5	0.5	1.0	1.0	0.9
1	1	2	1.0	1.0	1.0	1.0	1.0	1.0	1.0

Example: Detailed Calculation

Here, only consider the projection R_{12}:

$(x_1,$	$x_2,$	$x_3)$	$R(x_1, x_2, x_3)$	$R_{12}(x_1, x_2)$
0	0	0	0.4	
0	0	1	0.9	$(0,0) \mapsto \max[R(0, 0, 0), R(0, 0, 1), R(0, 0, 2)] = 0.9$
0	0	2	0.2	
0	1	0	1.0	
0	1	1	0.0	$(0,1) \mapsto \max[R(0, 1, 0), R(0, 1, 1), R(0, 1, 2)] = 1.0$
0	1	2	0.8	
1	0	0	0.5	
1	0	1	0.3	$(1,0) \mapsto \max[R(1, 0, 0), R(1, 0, 1), R(1, 0, 2)] = 0.5$
1	0	2	0.1	
1	1	0	0.0	
1	1	1	0.5	$(1,1) \mapsto \max[R(1, 1, 0), R(1, 1, 1), R(1, 1, 2)] = 1.0$
1	1	2	1.0	

Binary Fuzzy Relations

Representation and Inverse

Consider *e.g.* the **membership matrix** $\boldsymbol{R} = [r_{xy}]$ with $r_{xy} = R(x, y)$.

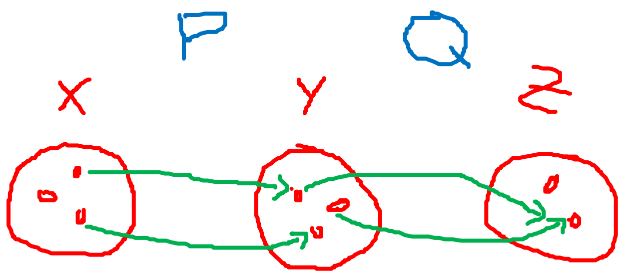
Its **inverse** $R^{-1}(Y, X)$ of $R(X, Y)$ is a relation on $Y \times X$ defined by

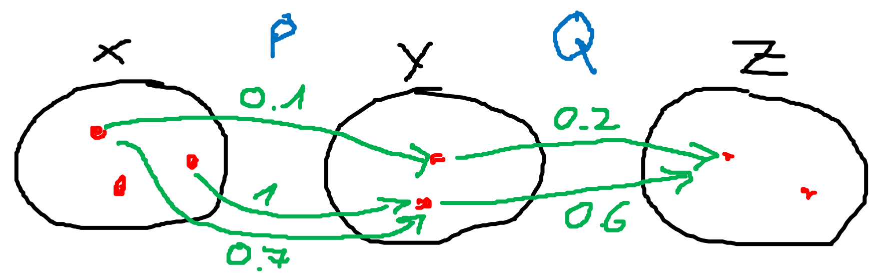
$$R^{-1}(y, x) = R(x, y) \quad \text{for all } x \in X,\ y \in Y.$$

$\boldsymbol{R}^{-1} = [r_{xy}^{-1}]$ representing $R^{-1}(y, x)$ is the transpose of $\boldsymbol{R}$ for $R(X, Y)$

$$(\boldsymbol{R}^{-1})^{-1} = \boldsymbol{R}$$

Standard Composition





Consider the binary relations $P(X,Y)$, $Q(Y,Z)$ with common set Y.

The **standard composition** of P and Q is defined as

$$(x,z) \in P \circ Q \iff \exists y \in Y : \{(x,y) \in P \land (y,z) \in Q\}.$$

In the fuzzy case this is generalized by

$$[P \circ Q](x,z) = \sup_{y \in Y} \{\min\{P(x,y),\ Q(y,z)\}\},\ \text{for all } x \in X \ \ and\ z \in Z.$$

If Y is finite, sup operator can be replaced by max.

The standard composition is also called **max-min composition**.

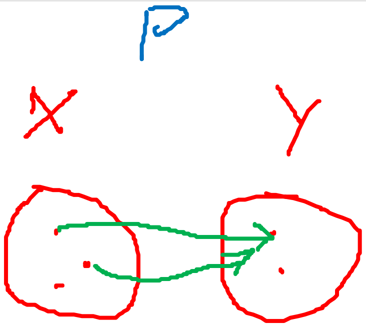
Example

$$P \circ Q = R$$

$$\begin{bmatrix} .3 & .5 & .8 \\ 0 & .7 & 1 \\ .4 & .6 & .5 \end{bmatrix} \circ \begin{bmatrix} .9 & .5 & .7 & .7 \\ .3 & .2 & 0 & .9 \\ 1 & 0 & .5 & .5 \end{bmatrix} = \begin{bmatrix} .8 & .3 & .5 & .5 \\ 1 & .2 & 5 & .7 \\ .5 & .4 & .5 & .5 \end{bmatrix}$$

$$\begin{aligned} r_{11} &= \max\{\min(p_{11}, q_{11}), \min(p_{12}, q_{21}), \min(p_{13}, q_{31})\} \\ &= \max\{\min(.3, .9), \min(.5, .3), \min(.8, 1)\} \\ &= .8 \\ r_{32} &= \max\{\min(p_{31}, q_{12}), \min(p_{32}, q_{22}), \min(p_{33}, q_{32})\} \\ &= \max\{\min(.4, .5), \min(.6, .2), \min(.5, 0)\} \\ &= .4 \end{aligned}$$

Inverse of Standard Composition



The inverse of the max-min composition follows from its definition:

$$[P(X,Y) \circ Q(Y,Z)]^{-1} = Q^{-1}(Z,Y) \circ P^{-1}(Y,X).$$

Its associativity also comes directly from its definition:

$$[P(X,Y)] \circ Q(Y,Z)] \circ R(Z,W) = P(X,Y) \circ [Q(Y,Z) \circ R(Z,W)].$$

Note that the standard composition is not commutative.

Matrix notation: $[r_{ij}] = [p_{ik}] \circ [q_{kj}]$ with $r_{ij} = \max_k \min(p_{ik}, q_{kj})$.

Example: Properties of Airplanes (Speed, Height, Type)

4 possible speeds:	s_1, s_2, s_3, s_4
3 heights:	h_1, h_2, h_3
2 types:	t_1, t_2

Consider the following fuzzy relations for airplanes:

- relation *A* between speed and height,
- relation *B* between height and the type.

$\boldsymbol{A}$	h_1	h_2	h_3
s_1	1	.2	0
s_2	.1	1	0
s_3	0	1	1
s_4	0	.3	1

$\boldsymbol{B}$	t_1	t_2
h_1	1	0
h_2	.9	1
h_3	0	.9

Binary Relations on a Single Set

It is also possible to define crisp or fuzzy binary relations among elements of a single set X.

Such a binary relation can be denoted by $R(X,X)$ or $R(X^2)$ which is a subset of $X \times X = X^2$.

These relations are often referred to as **directed graphs** which is also an representation of them.

- Each element of X is represented as node.
- Directed connections between nodes indicate pairs of $x \in X$ for which the grade of the membership is nonzero.
- Each connection is labeled by its actual membership grade of the corresponding pair in R.

Example

An example of $R(X, X)$ defined on $X = \{1, 2, 3, 4\}$.

Two different representation are shown below.

	1	2	3	4
1	.7	0	.3	0
2	0	.7	1	0
3	.9	0	0	1
4	0	0	.8	.5