OTTO VON GUERICKE UNIVERSITÄT MAGDEBURG

INF FAKULTÄT FÜR INFORMATIK

Fuzzy Control

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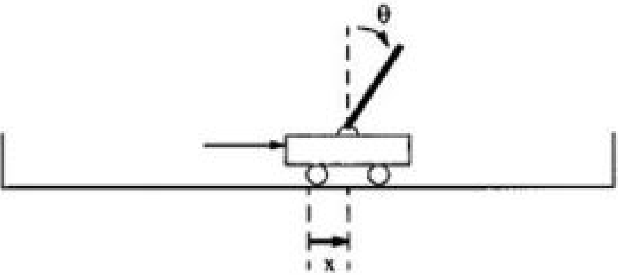
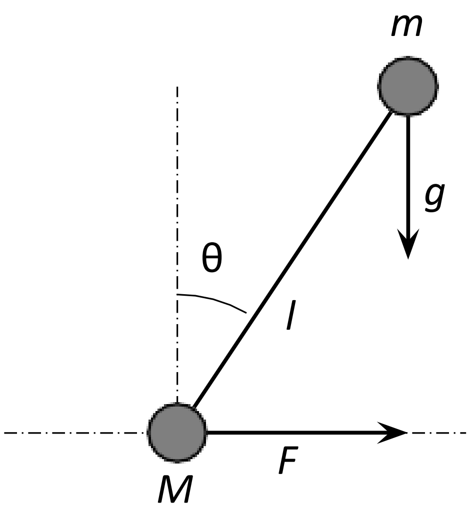
Stick Balancing

Inverted pendulum

Typical Example: Cartpole Problem





Balance an upright standing pole

Lower end of pole can be moved unrestrained along horizontal axis.

Mass *m* at foot and mass *M* at head.

Influence of mass of shaft itself is negligible.

Determine force *F* (control variable) that is necessary to balance pole standing upright.

That is measurement of following output variables:

- angle θ of pole in relation to vertical axis,
- change of angle, *i.e.* triangular velocity $\dot{\theta} = \frac{d\theta}{dt}$

Both should converge to zero.

Notation

Input variables $\xi_1, \ldots, \xi_n$, control variable η

Measurements: used to determine actual value of η

Assumption: ξ_i, $1 \leq i \leq n$ is value of X_i, $\eta \in Y$

Solution: **control function** φ

$$\begin{aligned} \varphi : X_1 \times \ldots \times X_n &\to Y \\ (x_1, \ldots, x_n) &\mapsto y \end{aligned}$$

Example: Cartpole Problem (cont.)

Angle $\theta \in X_1 = [-90°, 90°]$

Theoretically, every angle velocity $\dot{\theta}$ possible.

Extreme $\dot{\theta}$ are artificially achievable.

Assume $-45\,°/\text{s} \leq \dot{\theta} \leq 45\,°/\text{s}$ holds,
i.e. $\dot{\theta} \in X_2 = [-45\,°/\text{s}, 45\,°/\text{s}]$.

Absolute value of force $|F| \leq 10\,\mathrm{N}$.

Thus define $F \in Y = [-10\,\mathrm{N}, 10\,\mathrm{N}]$.

Example: Cartpole Problem (cont.)

Differential equation of cartpole problem:

$$(M+m)\sin^2\theta\cdot l\cdot\ddot{\theta}+m\cdot l\cdot\sin\theta\cos\theta\cdot\dot{\theta}^2-(M+m)\cdot g\cdot\sin\theta=-F\cdot\cos\theta$$

Compute $F(t)$ such that $\theta(t)$ and $\dot{\theta}(t)$ converge towards zero quickly.

Physical analysis demands knowledge about physical process.

In most real applications: No closed solution.
The standard is to use Runge Kutta Methods for systems of partial diffential equations for approximate solutions.
New successful methods are often nature inspired, such as Model-based Fuzzy Control, Reinforcement Learning, or evolutionary optimization techniques

Problems of Classical Approach

Often very difficult or even impossible to specify accurate mathematical model.

Description with differential equations is very complex.

Profound physical knowledge from engineer.

Exact solution can be very difficult.

Should be possible: to control process without physical-mathematical model. Human being knows how to balance a stick or to ride bike without knowing existence of differential equations.

Fuzzy Approach

Simulate behavior of human who knows how to control.

That is a **knowledge-based approach**.

Directly ask expert to perform analysis.

Then expert specifies knowledge as **linguistic rules**, *e.g.* for cartpole problem:

"If θ is approximately zero and $\dot{\theta}$ is also approximately zero, then F has to be approximately zero, too."

The aim is to find a simple solution that is **good enough.**

With further steps (using data and learning methods) the solutions is refined, if necessary.

Fuzzy Approach: Fuzzy Partitioning

1. Formulate set of linguistic rules:

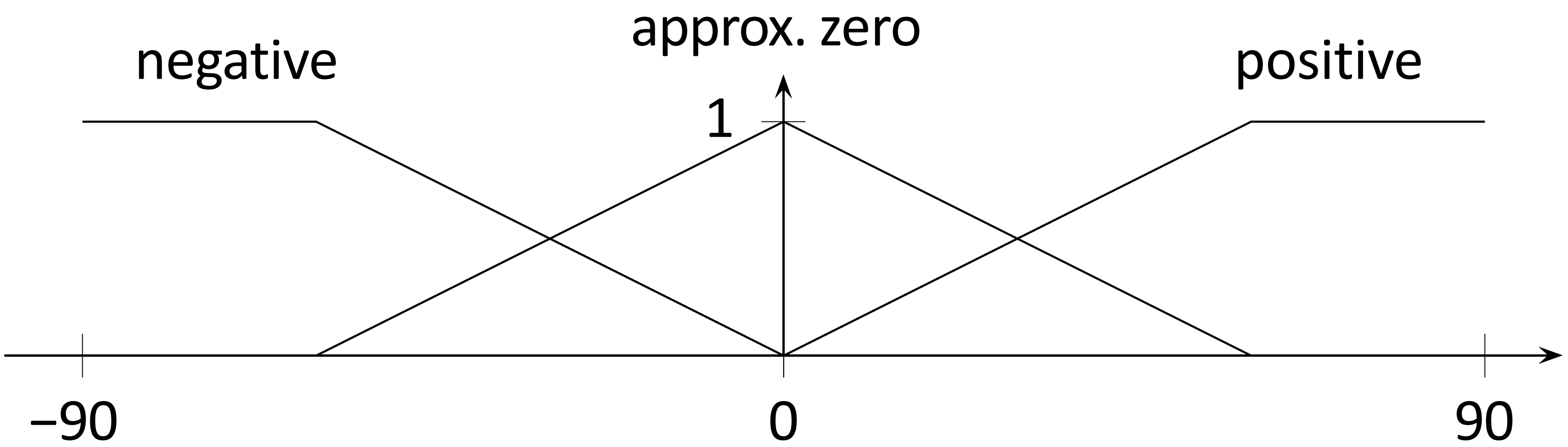
Determine linguistic terms (represented by fuzzy sets).

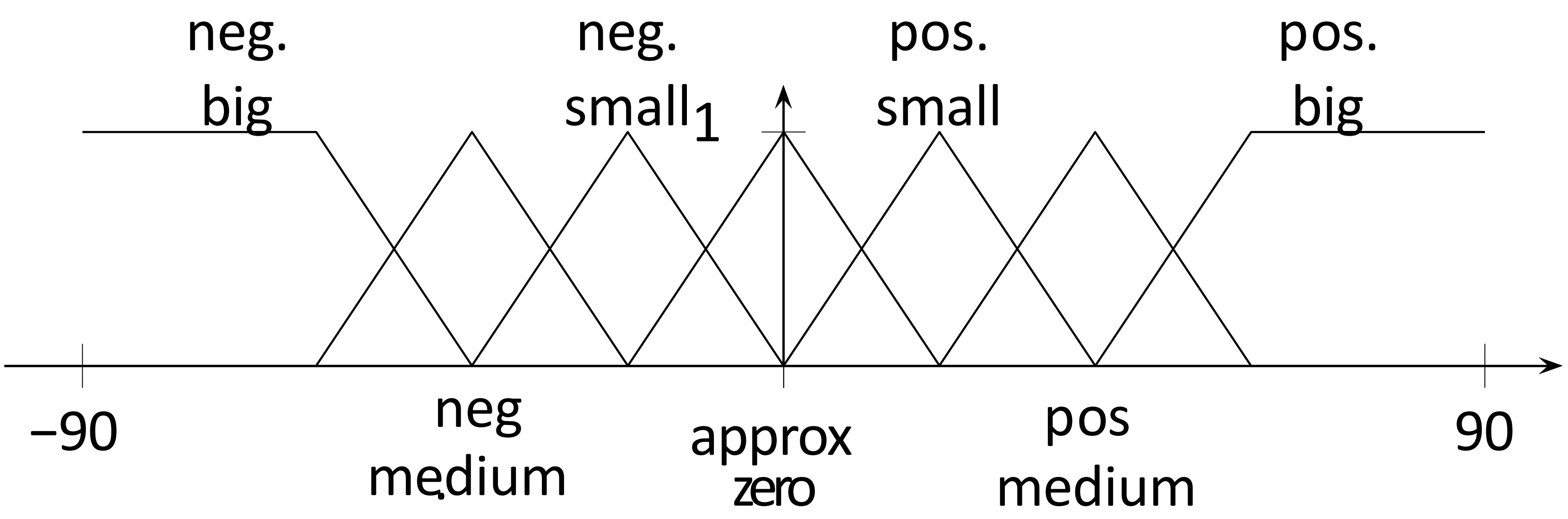
$X_1, \dots, X_n$ and Y is partitioned into fuzzy sets.

Define p_1 distinct fuzzy sets $\mu_1^{(1)}, \dots, \mu_{p_1}^{(1)} \in \mathcal{F}(X_1)$ on set X_1.

Associate linguistic term with each set.

Coarse and Fine Fuzzy Partitions





Fuzzy Approach: Fuzzy Partitioning II

X_1 corresponds to interval $[a, b]$ of real line,
$\mu_1^{(1)}, \ldots, \mu_{p_1}^{(1)} \in \mathcal{F}(X_1)$ are triangular functions

$$\mu_{x_0,\varepsilon} : [a, b] \to [0, 1]$$
$$x \mapsto 1 - \min\{\varepsilon \cdot |x - x_0|,\ 1\}.$$

If $a < x_1 < \ldots < x_{p_1} < b$, only $\mu_2^{(1)}, \ldots, \mu_{p_1-1}^{(1)}$ are triangular.

Boundaries are treated differently.

Fuzzy Approach: Fuzzy Partitioning III

left fuzzy set:

$$\mu_1^{(1)} : [a, b] \to [0, 1]$$
$$x \mapsto \begin{cases} 1, & \text{if } x \le x_1 \\ 1 - \min\{\varepsilon \cdot (x - x_1),\ 1\} & \text{otherwise} \end{cases}$$

right fuzzy set:

$$\mu_{p_1}^{(1)} : [a, b] \to [0, 1]$$
$$x \mapsto \begin{cases} 1, & \text{if } x_{p_1} \le x \\ 1 - \min\{\varepsilon \cdot (x_{p_1} - x),\ 1\} & \text{otherwise} \end{cases}$$

Example: Cartpole Problem (cont.)

X_1 partitioned into 7 fuzzy sets.

Similar fuzzy partitions for X_2 and Y.

Next step: specify rules

$$\text{if } \xi_1 \text{ is } A^{(1)} \text{ and } \ldots \text{ and } \xi_n \text{ is } A^{(n)} \text{ then } \eta \text{ is } B,$$

$A^{(1)}, \ldots, A^{(n)}$ and B represent linguistic terms corresponding to $\mu^{(1)}, \ldots, \mu^{(n)}$ and μ according to $X_1, \ldots, X_n$ and Y.

Rule base consists of k rules.

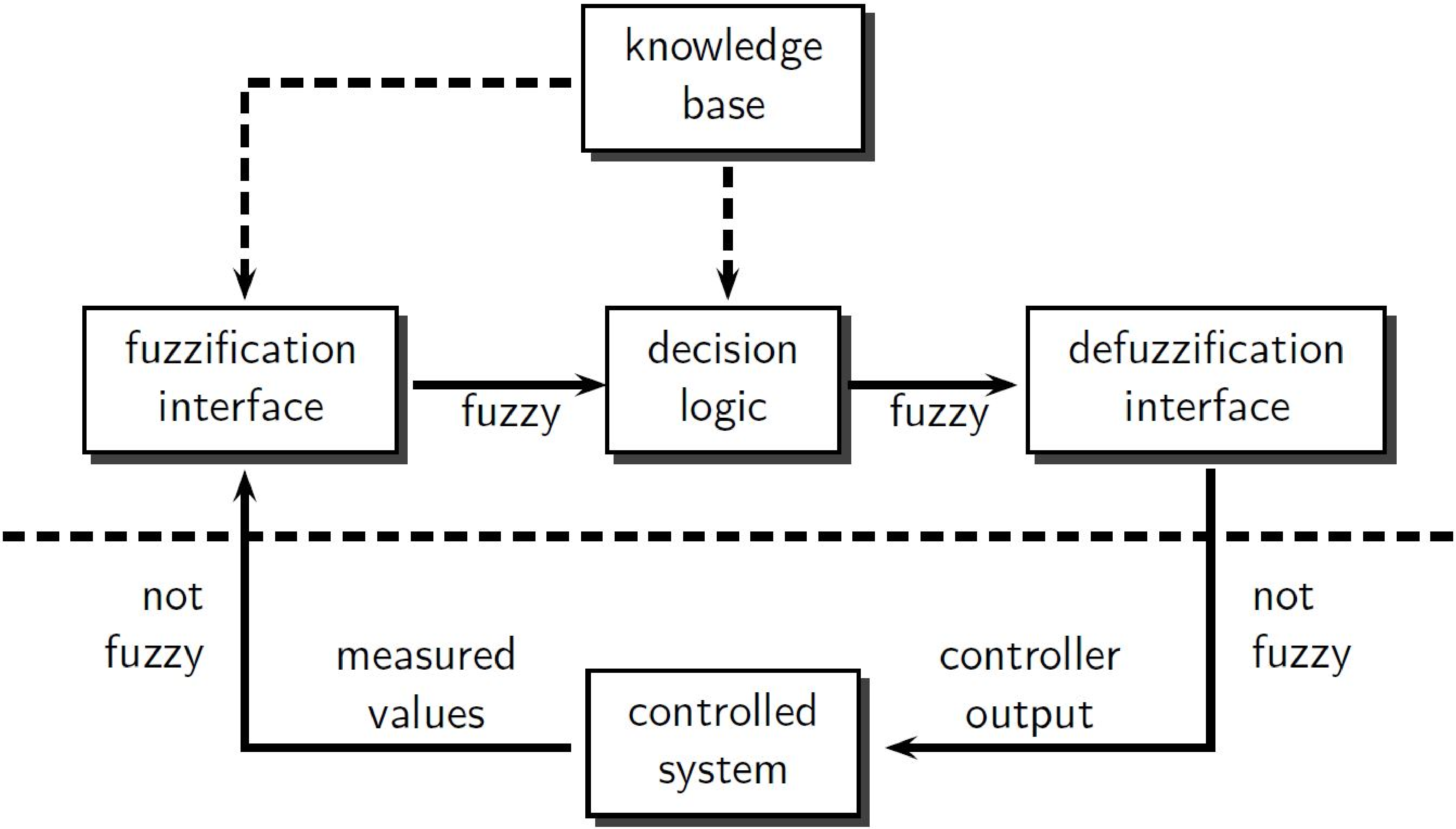
Example: Cartpole Problem (cont.)

$\dot{\theta}$ \ θ	nb	nm	ns	az	ps	pm	pb
nb			ps	pb			
nm				pm			
ns	nm		ns	ps			
az	nb	nm	ns	az	ps	pm	pb
ps				ns	ps		pm
pm				nm			
pb				nb	ns		

19 rules for cartpole problem, it is not necessary to determine all table entries. A table entry is interpreted as a rule:
If θ is *approximately zero* and $\dot{\theta}$ is *negative medium*
then *F* is *positive medium*.

Mamdani Controller

Architecture of a Mamdani Fuzzy Controller



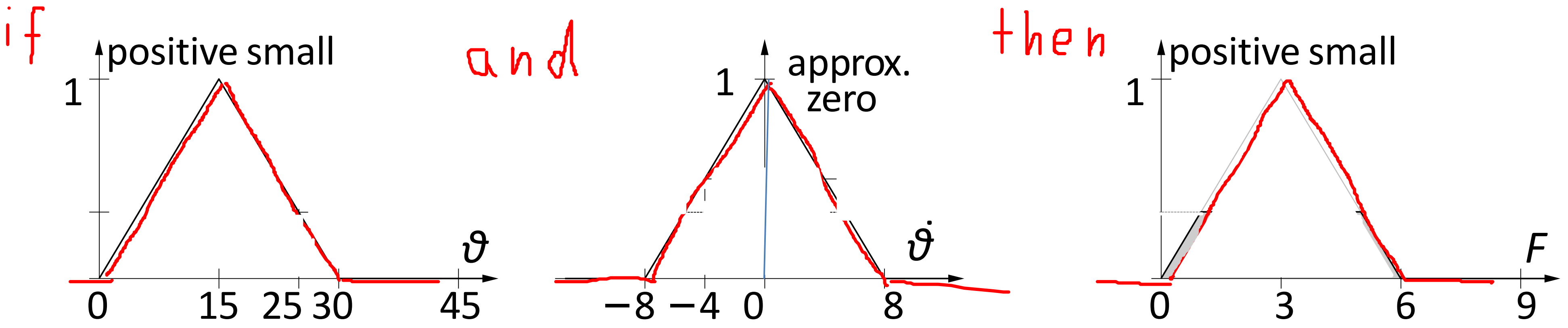
Qualitative Description of a Mamdani controller as a Rule System

ϑ

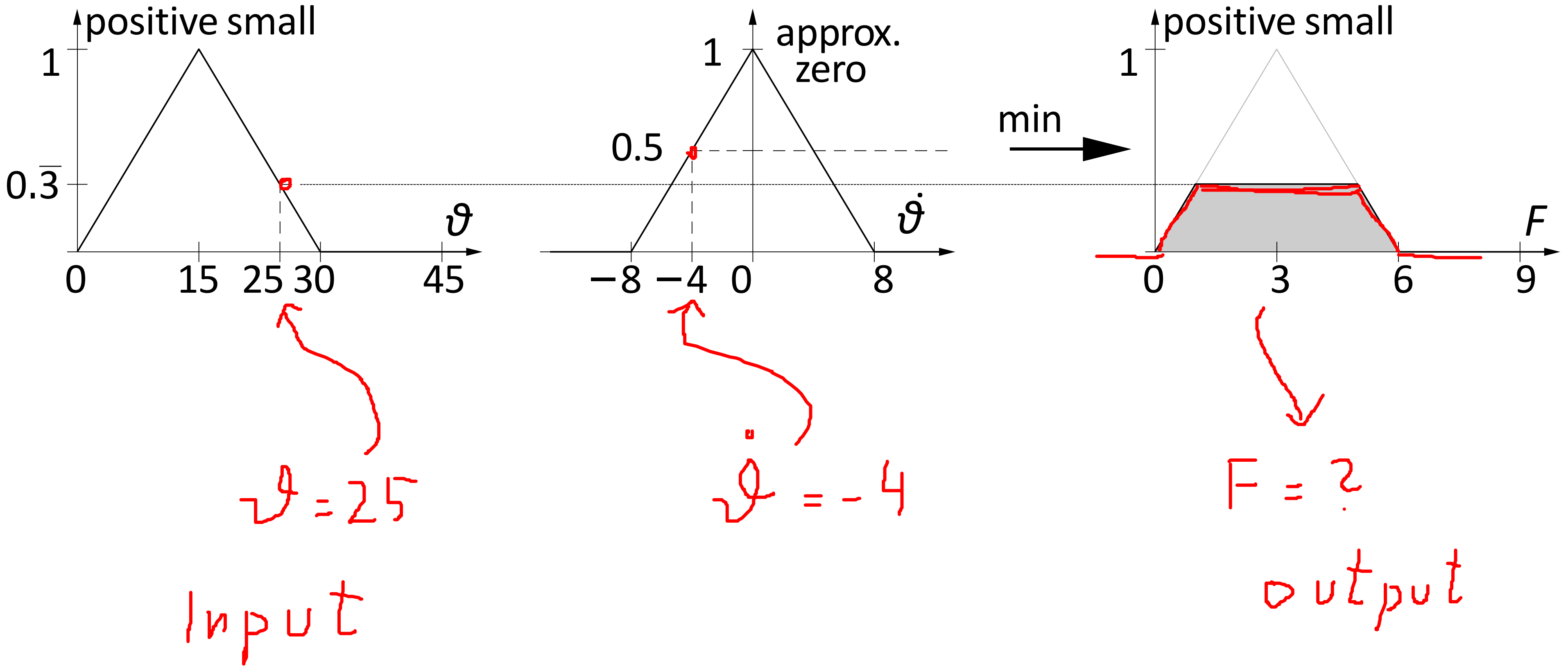
$\dot{\vartheta}$	nb	nm	ns	az	ps	pm	pb
nb			ps	pb			
nm				pm			
ns	nm		ns	ps			
az	nb	nm	ns	az	ps	pm	pb
ps				ns	ps		pm
pm				nm			
pb				nb	ns		

19 rules for cartpole problem:
If ϑ is *approximately zero* and $\dot{\vartheta}$ is *negative medium*
then *F* is *positive medium*.

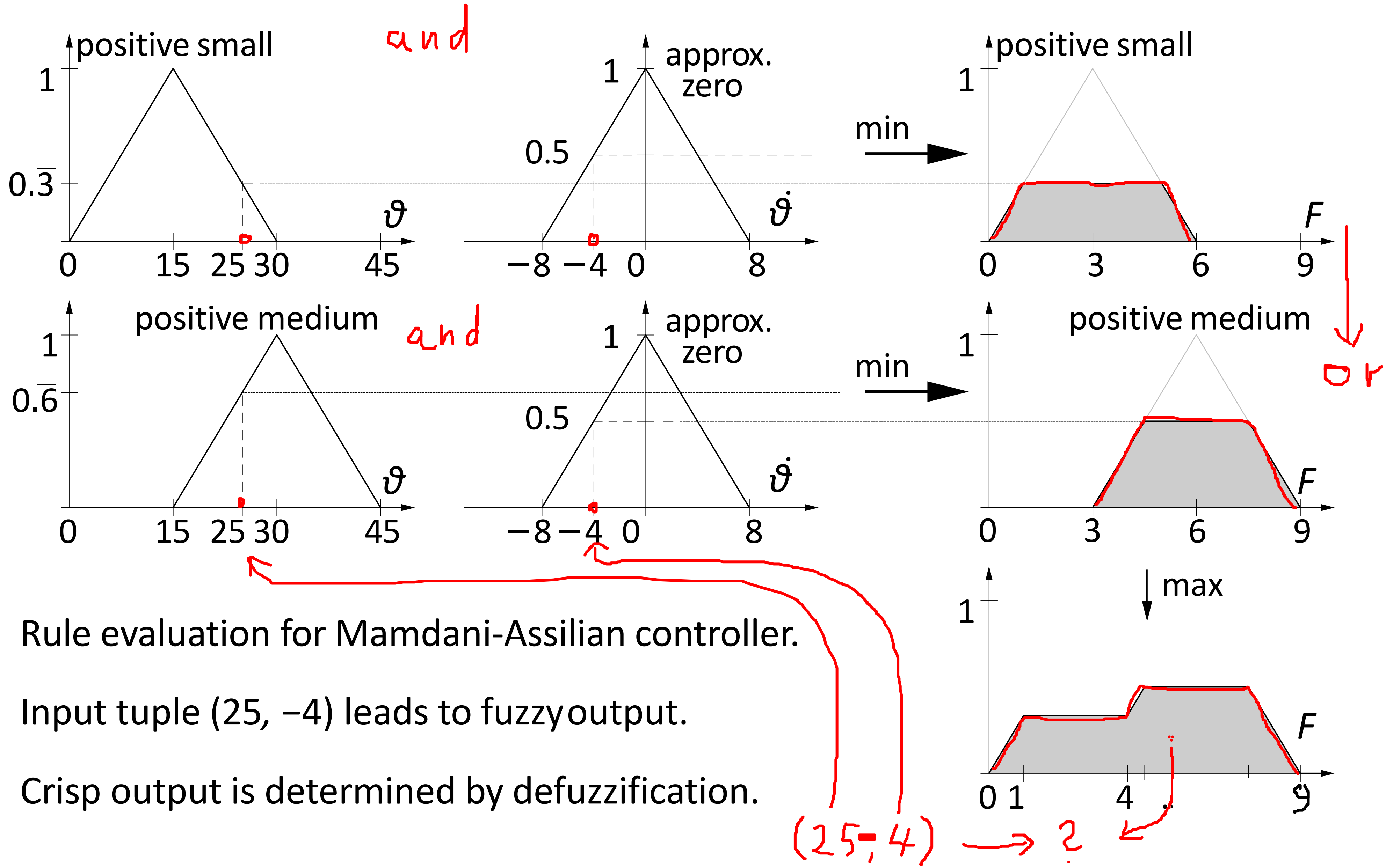
Evaluation of a single rule



Evaluation of a single rule



Evaluation of several rules



Rule evaluation for Mamdani-Assilian controller.

Input tuple (25, −4) leads to fuzzyoutput.

Crisp output is determined by defuzzification.

Definition of Table-based Control Function I

Given is the measurement $(x_1, \dots, x_n) \in X_1 \times \dots \times X_n$
Consider a rule R

$$\text{if } \mu^{(1)} \text{ and } \dots \text{ and } \mu^{(n)} \text{ then } \eta.$$

The fuzzyfication unit computes for the input $(x_1, \dots, x_n)$ a „degree of fulfillment“ of the premise of the rule:

For $1 \le v \le n$, the membership degree $\mu^{(v)}(x_v)$ is calculated. The n degrees are combined conjunctively with the min-operator and give the fulfillment degree α

For each rule R_r with $1 \le r \le k$, compute the fulfillment degree α_r

Definition of Table-based Control Function II

For the input $(x_1, \ldots, x_n)$ and a rule R the decision unit calculates the output

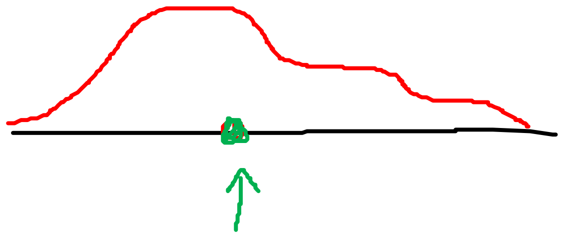
$$\mu^{\text{output}(R)}_{x_1,\ldots,x_n} : Y \to [0, 1] ,$$

$$y \mapsto \min \left(\mu^{(1)}(x_1), \ldots, \mu^{(n)}(x_n), \eta(y) \right) .$$

Definition of Table-based Control Function III

The decision logic combines the output fuzzy sets from all rules $R_1,\ldots,R_k$ by using the or-operator **maximum.** This results in the output fuzzy set

$$\mu^{\text{output}}_{x_1,\ldots,x_n} : Y \to [0, 1]$$

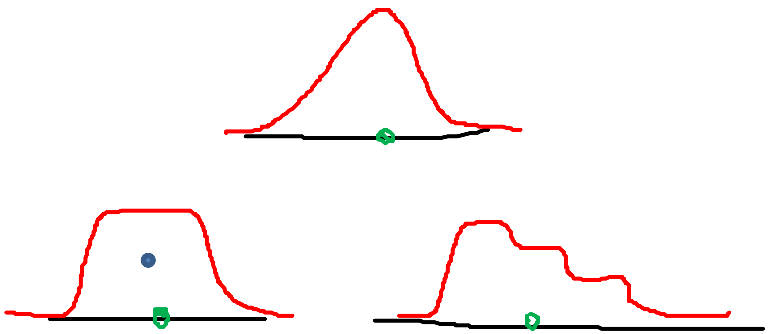


Then $\mu^{\text{output}}_{x_1,\ldots,x_n}$ is passed to defuzzification interface.

Definition of Table-based Control Function IV

Defuzzification interface derives crisp value from $\mu^{\text{output}}_{x_1,\ldots,x_n}$.

Most common **defuzzification** methods:



- max criterion,
- mean of maxima,
- center of gravity.

See Google Patents at defuzzification : More than 1080 methods

Center of Gravity (COG) Method

Same preconditions as MOM method.

η = center of gravity/area of $\mu^{\text{output}}_{x_1,\ldots,x_n}$

If Y is finite, then

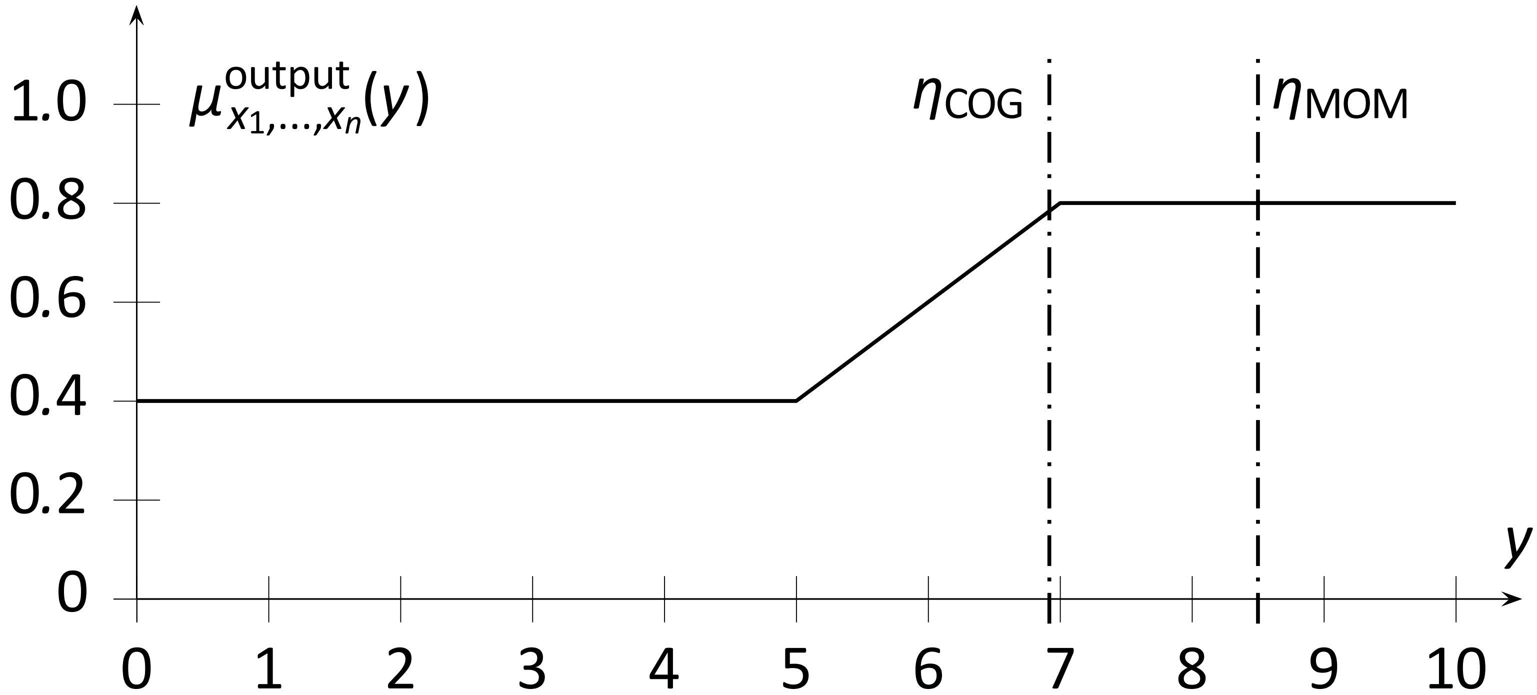
$$\eta = \frac{\sum_{y_i \in Y} y_i \cdot \mu^{\text{output}}_{x_1,\ldots,x_n}(y_i)}{\sum_{y_i \in Y} \mu^{\text{output}}_{x_1,\ldots,x_n}(y_i)}.$$

If Y is infinite, then

$$\eta = \frac{\int_{y \in Y} y \cdot \mu^{\text{output}}_{x_1,\ldots,x_n}(y)\, dy}{\int_{y \in Y} \mu^{\text{output}}_{x_1,\ldots,x_n}(y)\, dy}.$$

Example

Task: compute η_{COG} and η_{MOM} of fuzzy set shown below.



Example for COG

Continuous and Discrete Output Space

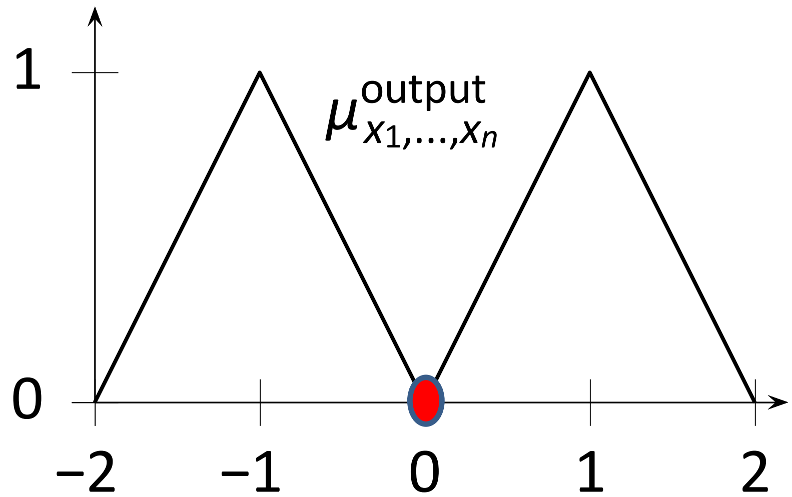
$$\eta_{\mathrm{COG}} = \frac{\int_0^{10} y \cdot \mu^{\mathrm{output}}_{x_1,\ldots,x_n}(y)\, dy}{\int_0^{10} \mu^{\mathrm{output}}_{x_1,\ldots,x_n}(y)\, dy}$$

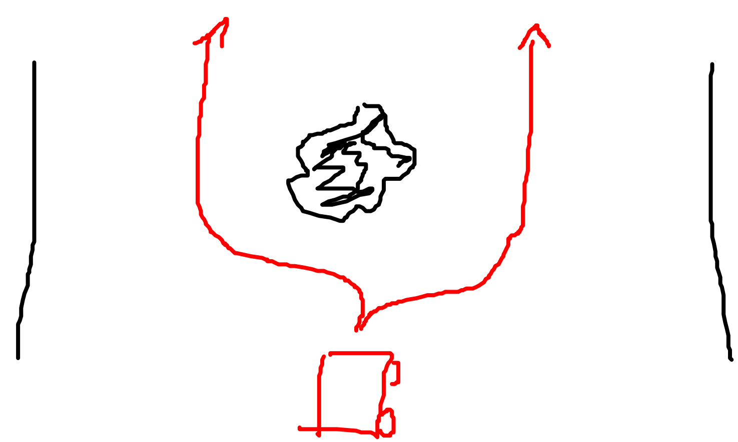
$$= \frac{\int_0^5 0.4y\, dy + \int_5^7 (0.2y - 0.6)y\, dy + \int_7^{10} 0.8y\, dy}{5 \cdot 0.4 + 2 \cdot \frac{0.8+0.4}{2} + 3 \cdot 0.8}$$

$$\approx \frac{38.7333}{5.6} \approx 6.917$$

Problem Cases for MOM and COG

Stone on the street, the car should not use COG (giving 0) but -1 or +1

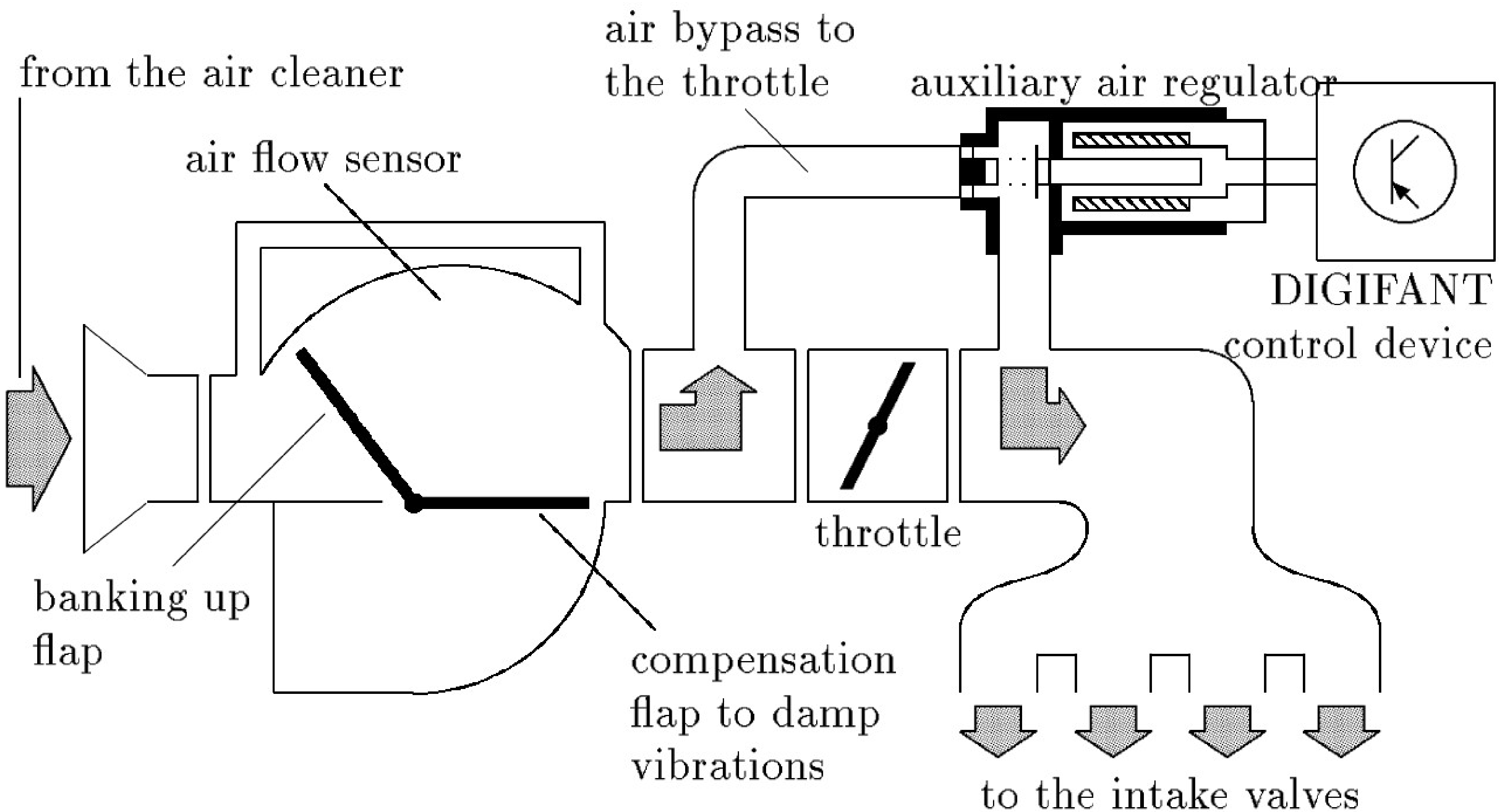


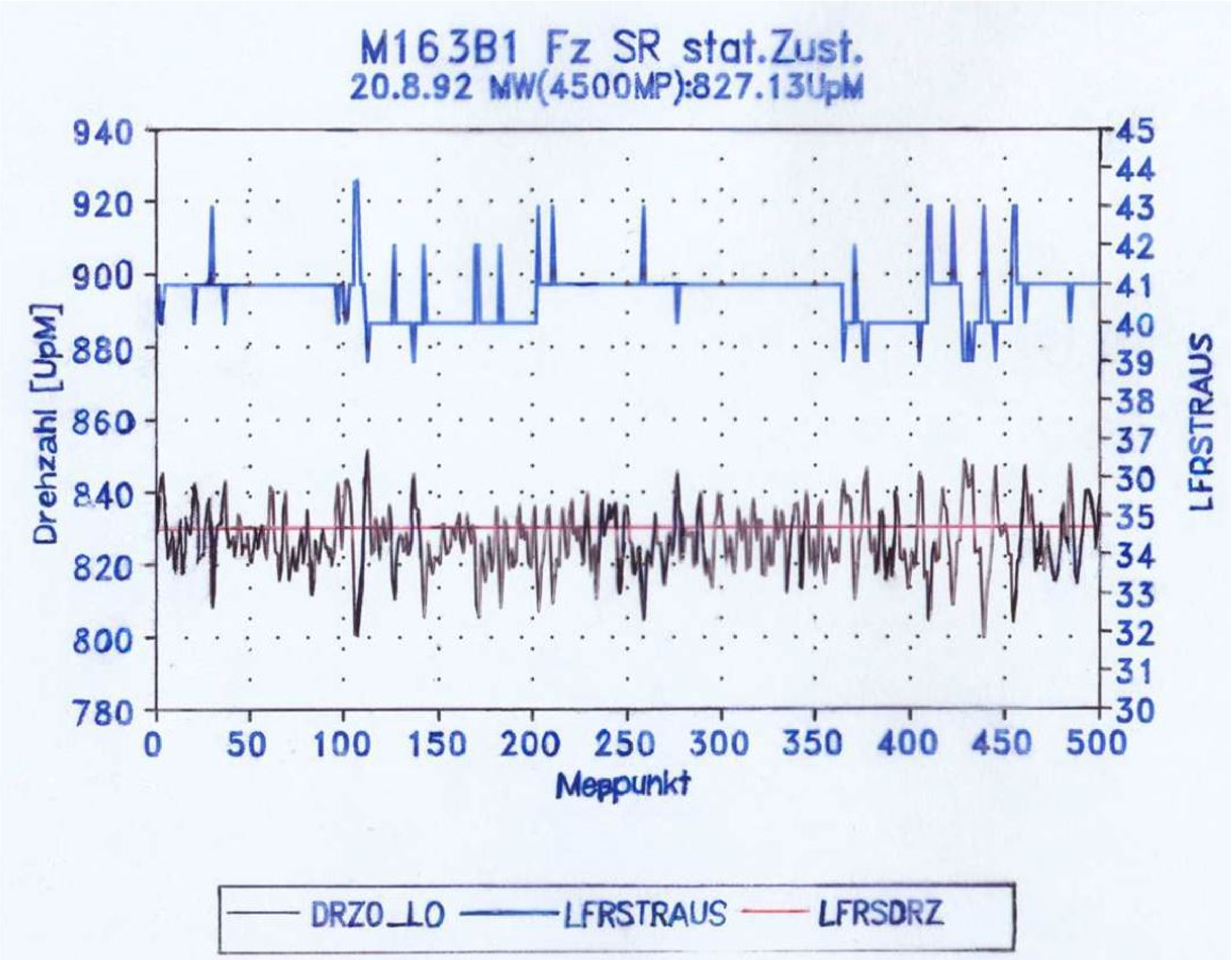


Mamdani Control Applications

Example: Engine Idle Speed Control

VW 2000cc 116hp Motor (Golf GTI)

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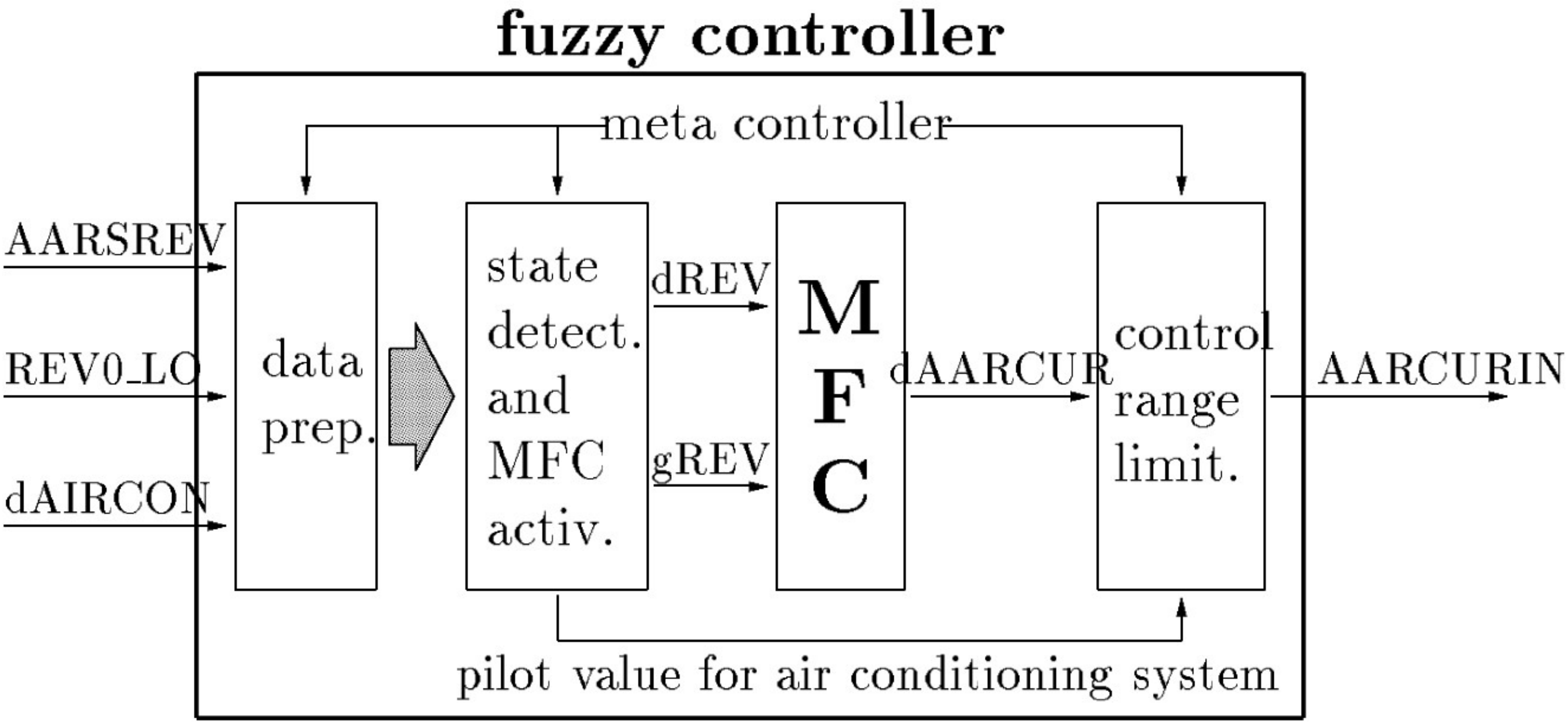
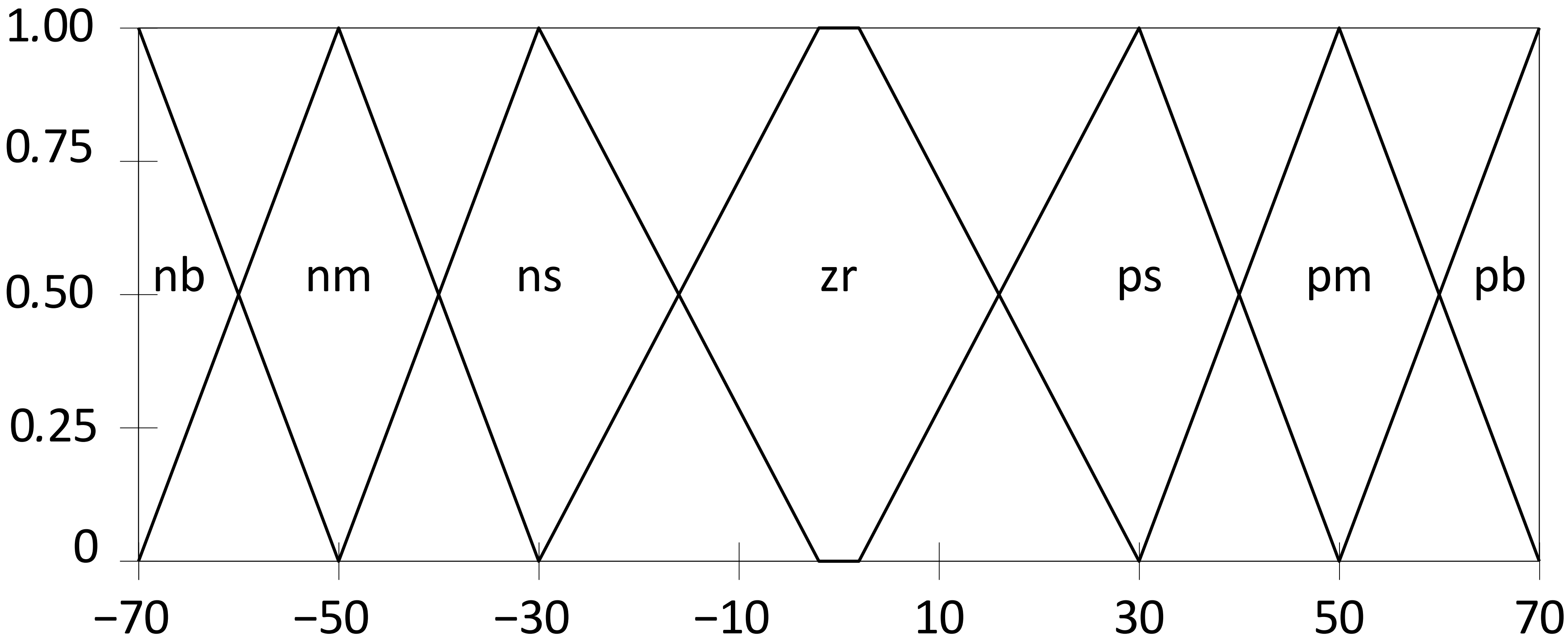
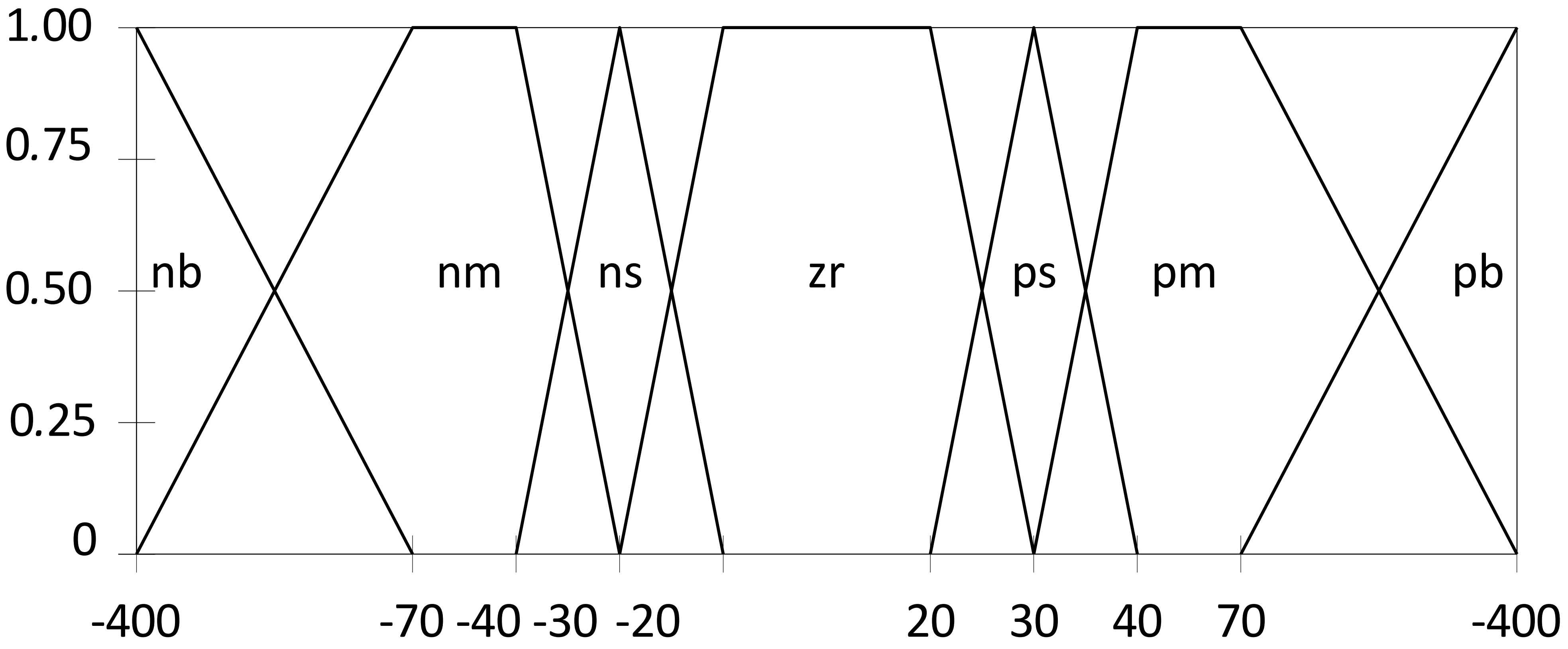
Structure of the Fuzzy Controller



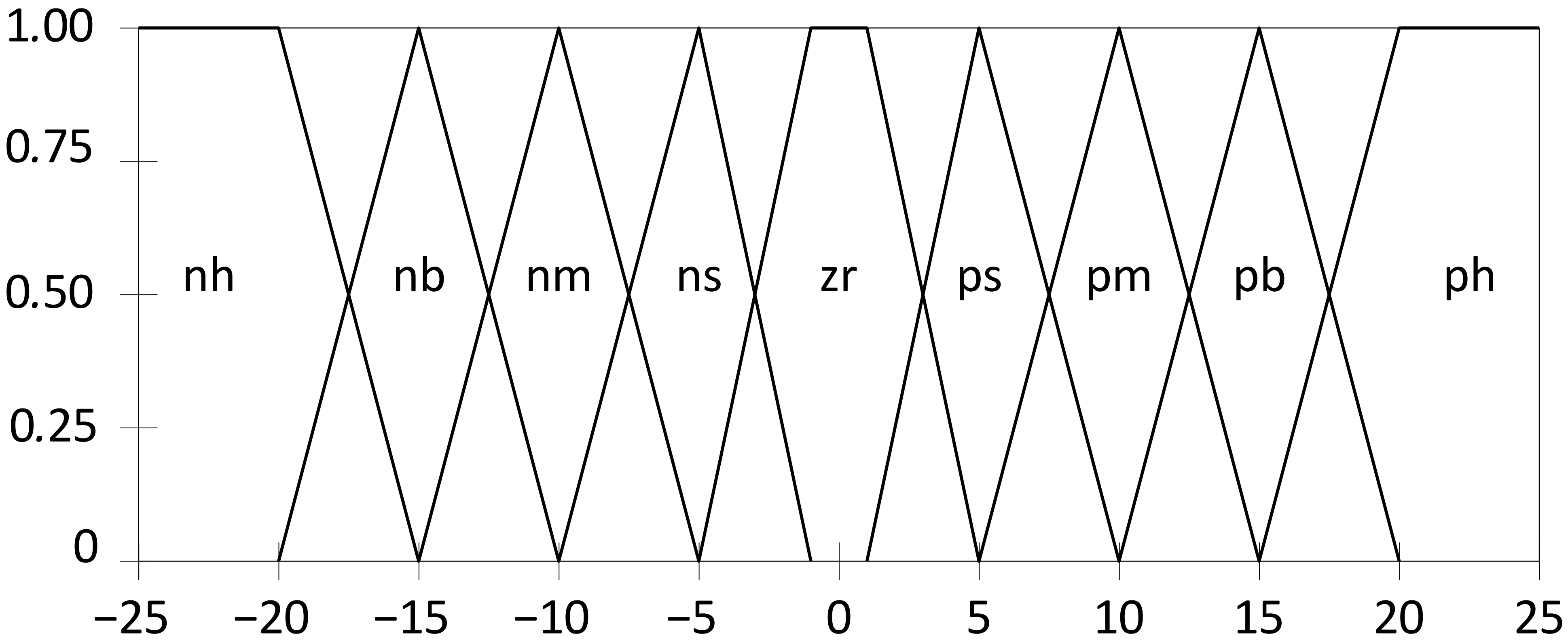
Deviation of the Number of Revolutions dREV



Gradient of the Number of Revolutions gREV



Change of Current for Auxiliary Air Regulator dAARCUR



Rule Base

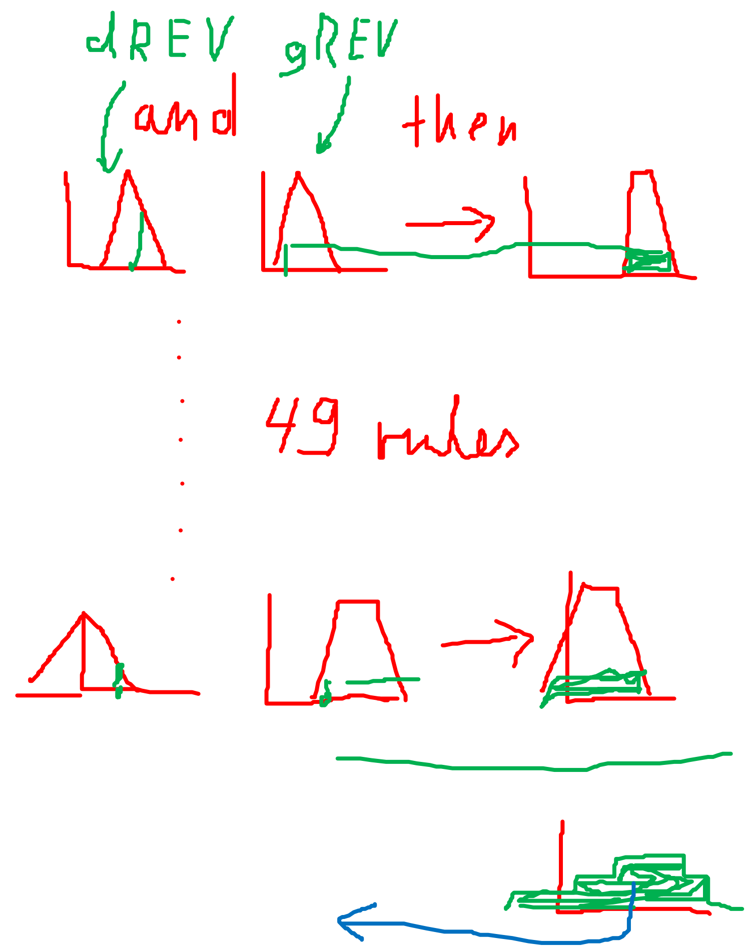
If the deviation from the desired number of revolutions is negative small **and** the gradient is negative medium,
then the change of the current for the auxiliary air regulation should be positive medium.

		gREV						
		nb	nm	ns	az	ps	pm	pb
	nb	ph	pb	pb	pm	pm	ps	ps
	nm	ph	pb	pm	pm	ps	ps	az
	ns	pb	pm	ps	ps	az	az	az
dREV	az	ps	ps	az	az	az	nm	ns
	ps	az	az	az	ns	ns	nm	nb
	pm	az	ns	ns	ns	nb	nb	nh
	pb	ns	ns	nm	nb	nb	nb	nh

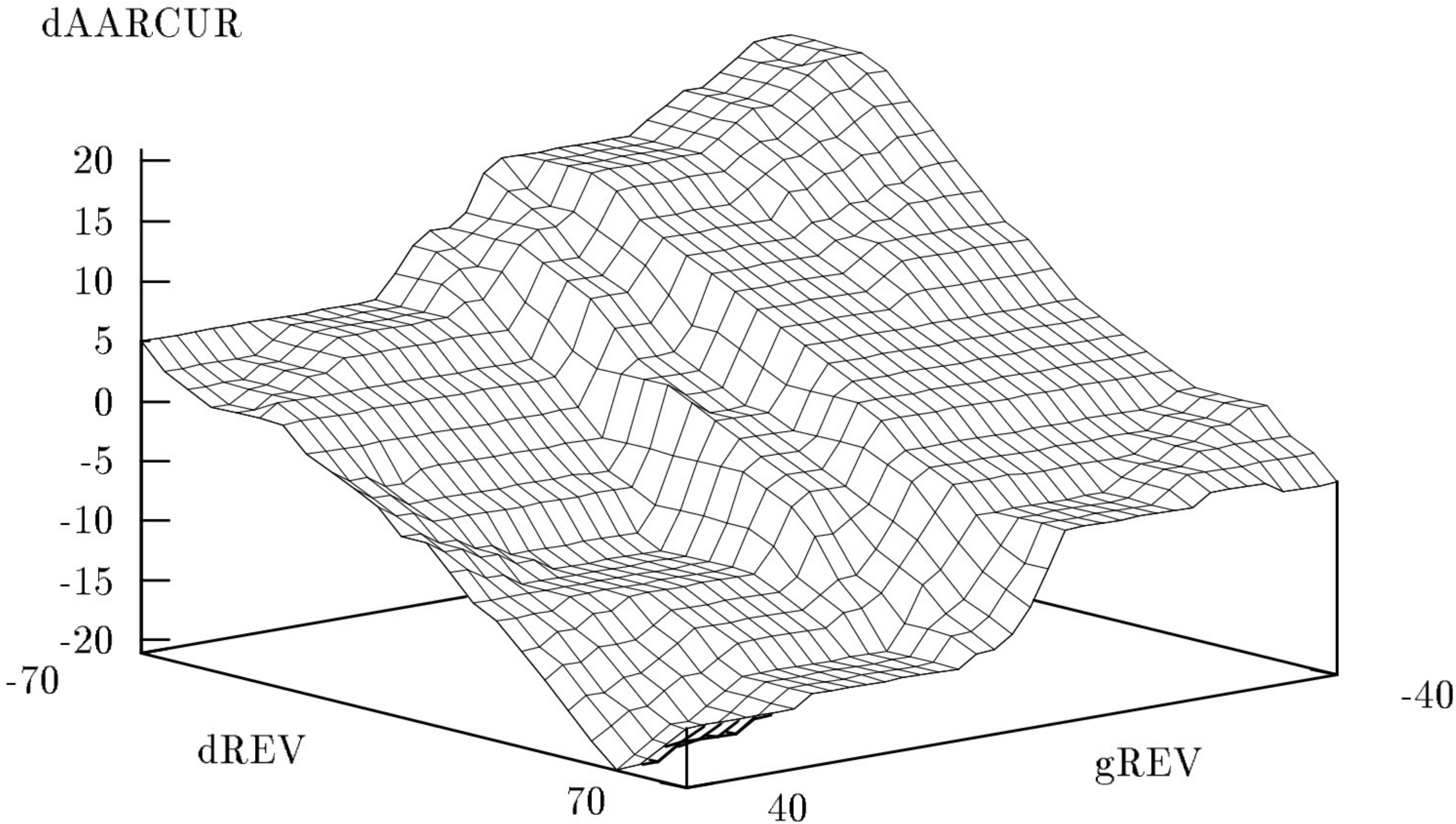
Rule Base

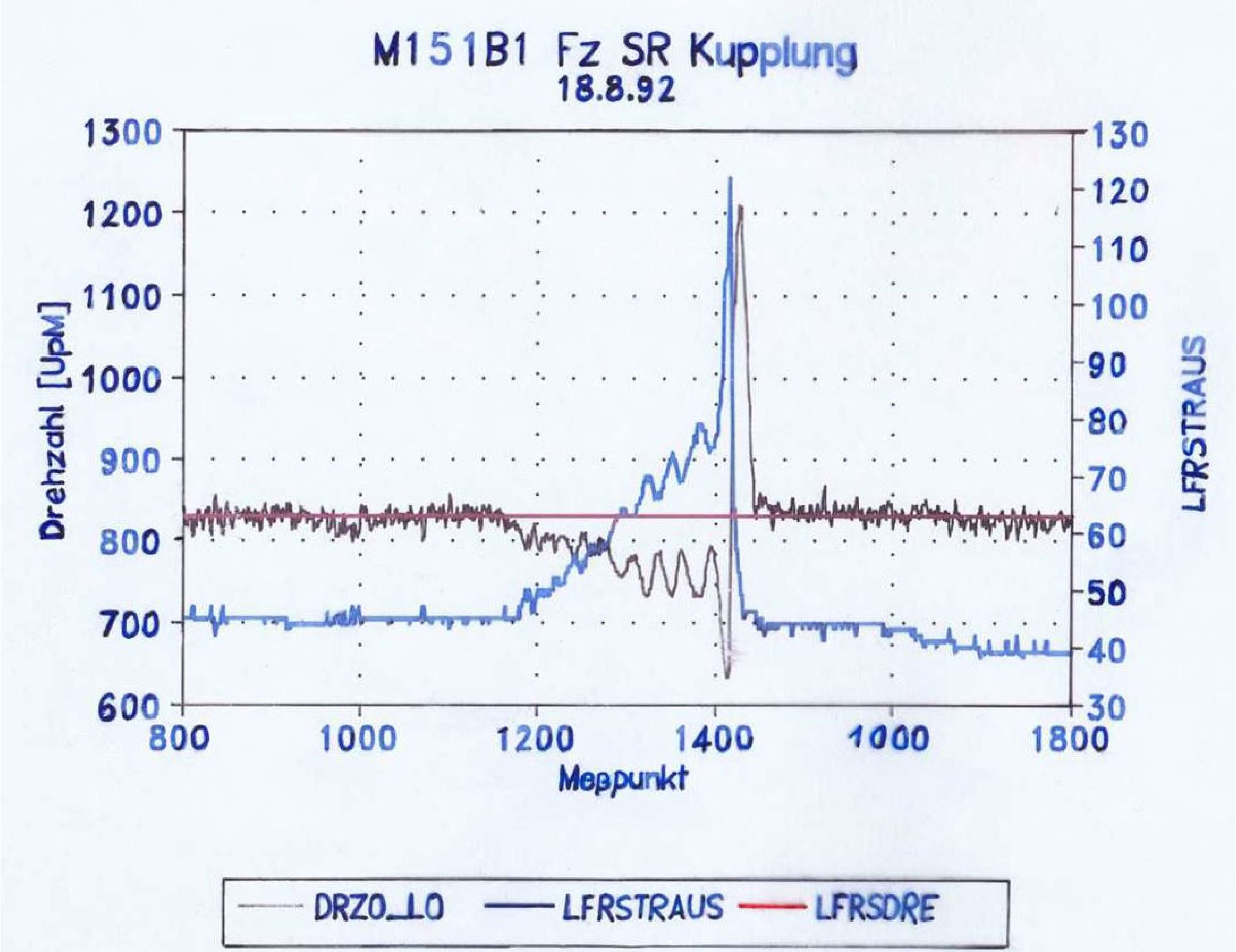
If the deviation from the desired number of revolutions is negative small **and** the gradient is negative medium,
then the change of the current for the auxiliary air regulation should be positive medium.

		gREV						
		nb	nm	ns	az	ps	pm	pb
	nb	ph	pb	pb	pm	pm	ps	ps
	nm	ph	pb	pm	pm	ps	ps	az
	ns	pb	pm	ps	ps	az	az	az
dREV	az	ps	ps	az	az	az	nm	ns
	ps	az	az	az	ns	ns	nm	nb
	pm	az	ns	ns	ns	nb	nb	nh
	pb	ns	ns	nm	nb	nb	nb	nh



Performance Characteristics





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Drehzahl [UpM]
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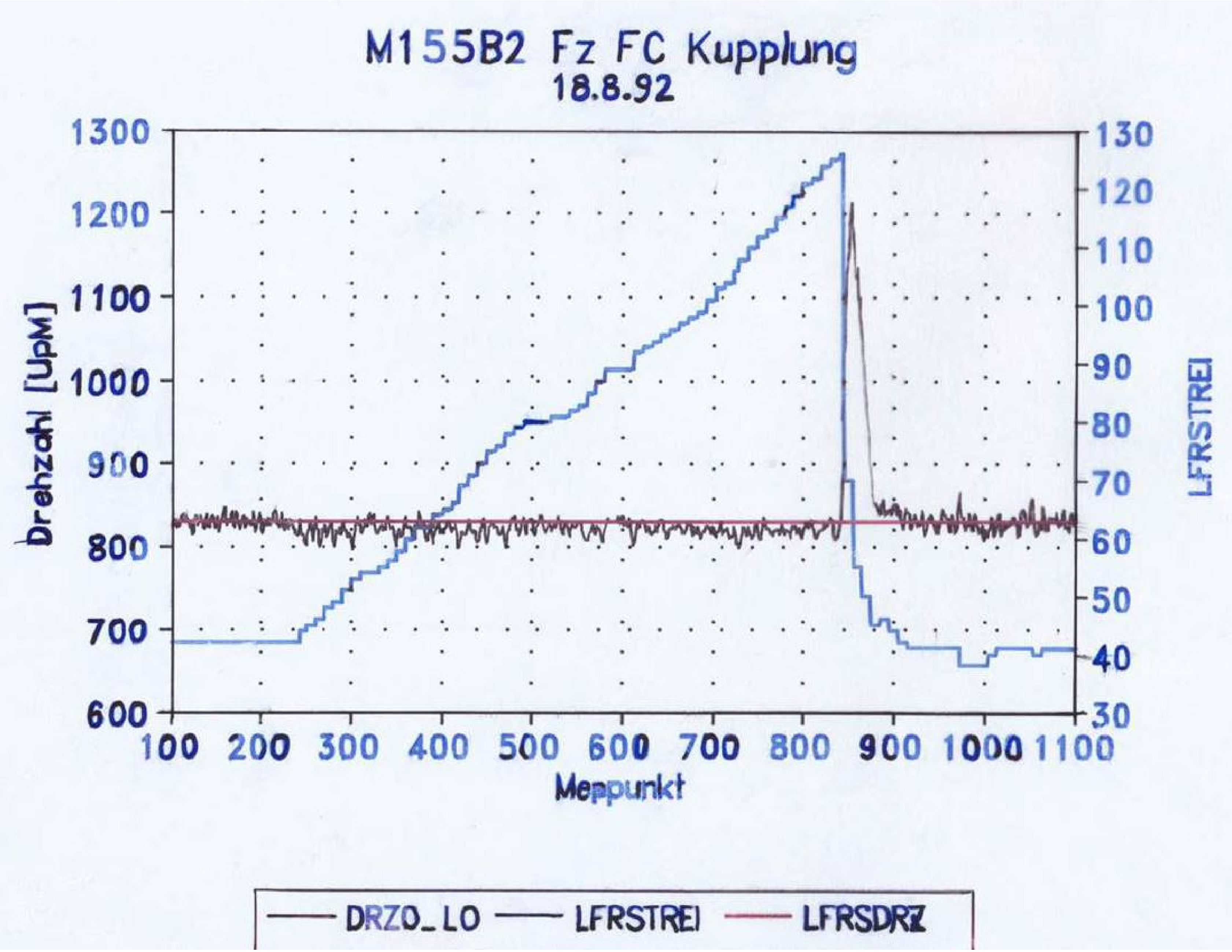
M155B2 Fz FC Kupplung
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Drehzahl [UpM]
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Meßpunkt
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LFRSTREI
LFRSDRZ

Example: Automatic Gear Box AG4

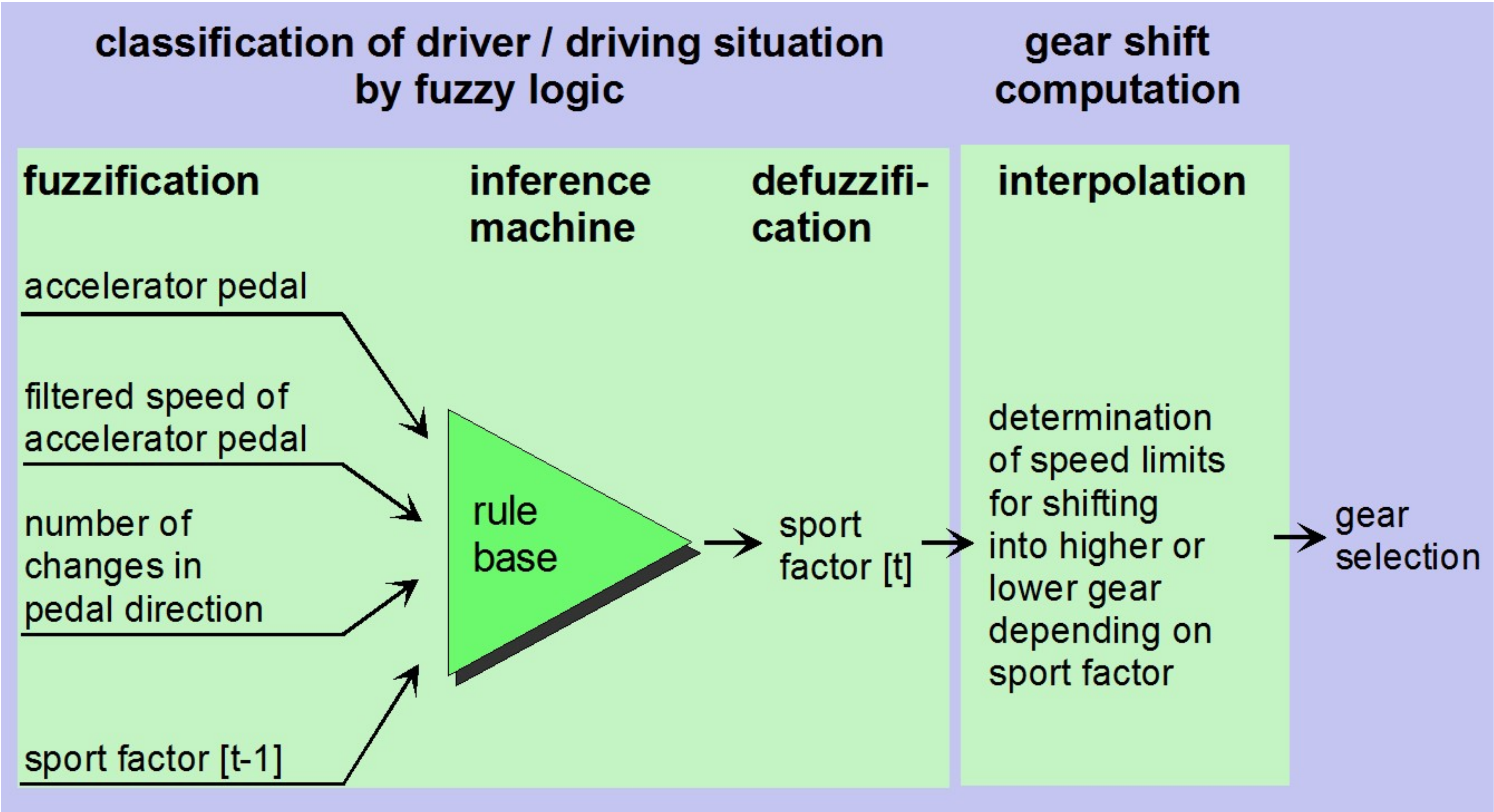
Idea: car “watches” driver and classifies him/her into calm, normal, sportive (assign sport factor [0, 1]), or nervous (calm down driver).

Test car: different drivers, classification by expert (passenger).

Simultaneous measurement of 14 attributes, *e.g.* , speed, position of accelerator pedal, speed of accelerator pedal, kick down, steering wheel angle.

Example: Automatic Gear Box

Continuously Adapting Gear Shift Schedule in VW New Beetle



Example: Automatic Gear Box

Technical Details

Optimized program on Digimat:

24 byte RAM

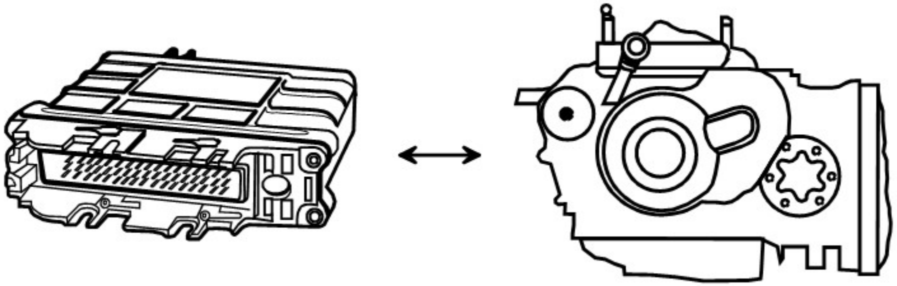
702 byte ROM

uses 7 Mamdani fuzzy rules

Runtime: 80ms

12 times per second new sport factor is assigned.

In Series Line



Takagi Sugeno Control

Takagi-Sugeno Controller

Proposed by Tomohiro Takagi and Michio Sugeno.

Modification/extension of Mamdani controller.

Both in common: fuzzy partitions of input domain $X_1, \dots, X_n$.

Difference to Mamdani controller:

- no fuzzy partition of output domain Y, no defuzzification
- controller rules $R_1, \dots, R_k$ are given by

$$R_r : \textbf{if } \xi_1 \text{ is } A^{(1)}_{i_{1,r}} \text{ and } \dots \text{ and } \xi_n \text{ is } A^{(n)}_{i_{n,r}}$$
$$\textbf{then } \eta_r = f_r(\xi_1, \dots, \xi_n),$$

$f_r : X_1 \times \dots \times X_n \to Y.$

- Typically, f_r is linear, *i.e.* $f_r(x_1, \dots, x_n) = a_0^{(r)} + \sum_{i=1}^{n} a_i^{(r)} x_i$.

Takagi-Sugeno Controller: Conclusion

For given input $(x_1, \ldots, x_n)$ and for each R_r, decision logic computes truth value α_r of each premise, and then $f_r(x_1, \ldots, x_n)$.

Analogously to Mamdani controller:

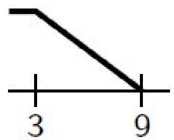
$$\alpha_r = \min\left\{\mu^{(1)}_{i_{1,r}}(x_1), \ldots, \mu^{(n)}_{i_{n,r}}(x_n)\right\}.$$

Output equals crisp control value

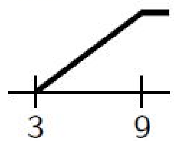
$$\eta = \frac{\sum_{r=1}^{k} \alpha_r \cdot f_r(x_1, \ldots, x_n)}{\sum_{r=1}^{k} \alpha_r}.$$

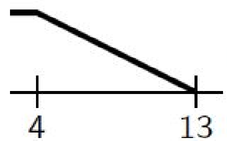
Thus no defuzzification method necessary.

Example

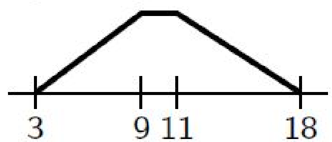
R_1 : if ξ_1 is 

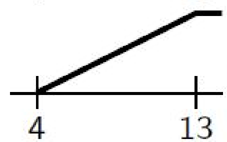
then $\eta_1 = 1 \cdot \xi_1 + 0.5 \cdot \xi_2 + 1$

R_2 : if ξ_1 is 

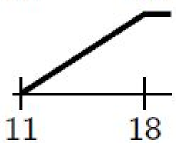
and ξ_2 is 

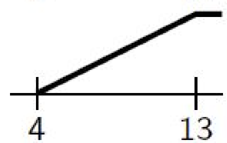
then $\eta_2 = -0.1 \cdot \xi_1 + 4 \cdot \xi_2 + 1.2$

R_3 : if ξ_1 is 

and ξ_2 is 

then $\eta_3 = 0.9 \cdot \xi_1 + 0.7 \cdot \xi_2 + 9$

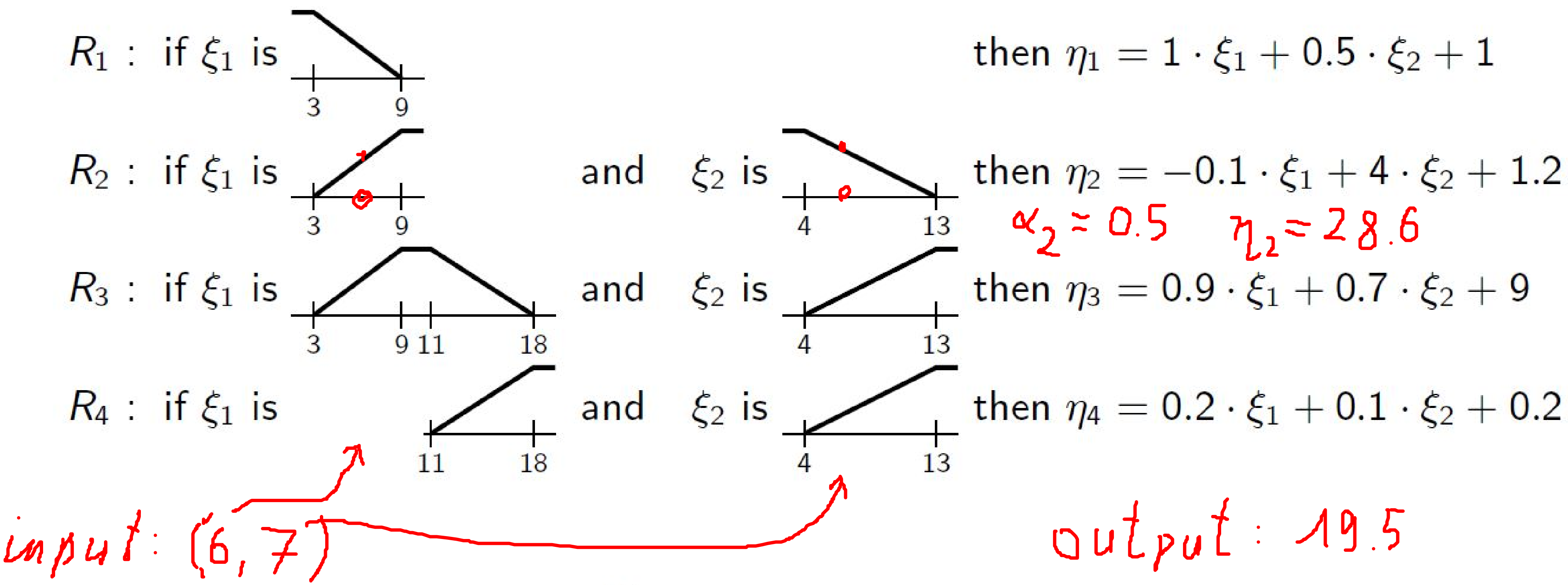
R_4 : if ξ_1 is 

and ξ_2 is 

then $\eta_4 = 0.2 \cdot \xi_1 + 0.1 \cdot \xi_2 + 0.2$

If a certain clause "x_j is $A^{(j)}_{i_{j,r}}$" in rule R_r is missing, then $\mu_{i_{j,r}}(x_j) \equiv 1$ for all linguistic values $i_{j,r}$.

For instance, here x_2 in R_1, so $\mu_{i_{2,1}}(x_2) \equiv 1$ for all $i_{2,1}$.



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Example: Output Computation

input: $(\xi_1, \xi_2) = (6, 7)$

$\alpha_1 = {}^1\!/_2 \wedge 1 = {}^1\!/_2$

$\alpha_2 = {}^1\!/_2 \wedge {}^2\!/_3 = {}^1\!/_2$

$\alpha_3 = {}^1\!/_2 \wedge {}^1\!/_3 = {}^1\!/_3$

$\alpha_4 = 0 \wedge {}^1\!/_3 = 0$

$\eta_1 = 6 + {}^7\!/_2 + 1 = 10.5$

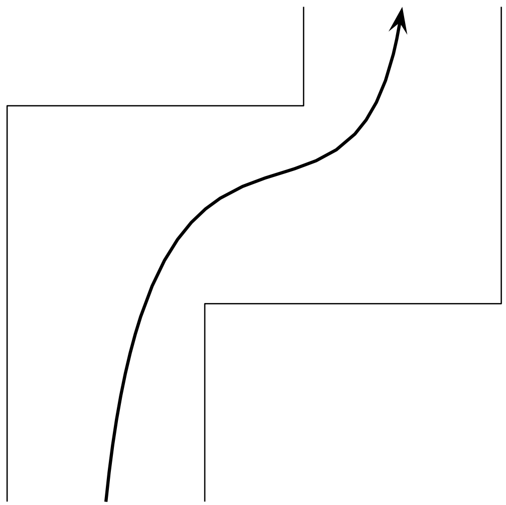
$\eta_2 = -0.6 + 28 + 1.2 = 28.6$

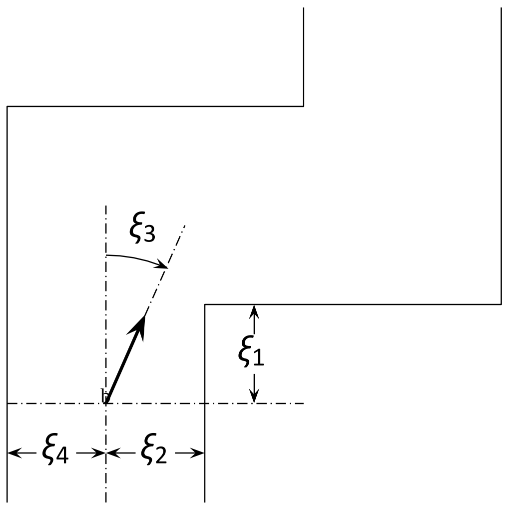
$\eta_3 = 0.9 \cdot 6 + 0.7 \cdot 7 + 9 = 19.3$

$\eta_4 = 6 + {}^7\!/_2 + 1 = 10.5$

output: $$\eta = f(6, 7) = \frac{{}^1\!/_2 \cdot 10.5 + {}^1\!/_2 \cdot 28.6 + {}^1\!/_3 \cdot 19.3}{{}^1\!/_2 + {}^1\!/_2 + {}^1\!/_3} = 19.5$$

Example: Passing a Bend





Pass a bend with a car at constant speed.

Measured inputs:

ξ_1 : distance of car to beginning of bend

ξ_2 : distance of car to inner barrier

ξ_3 : direction (angle) of car

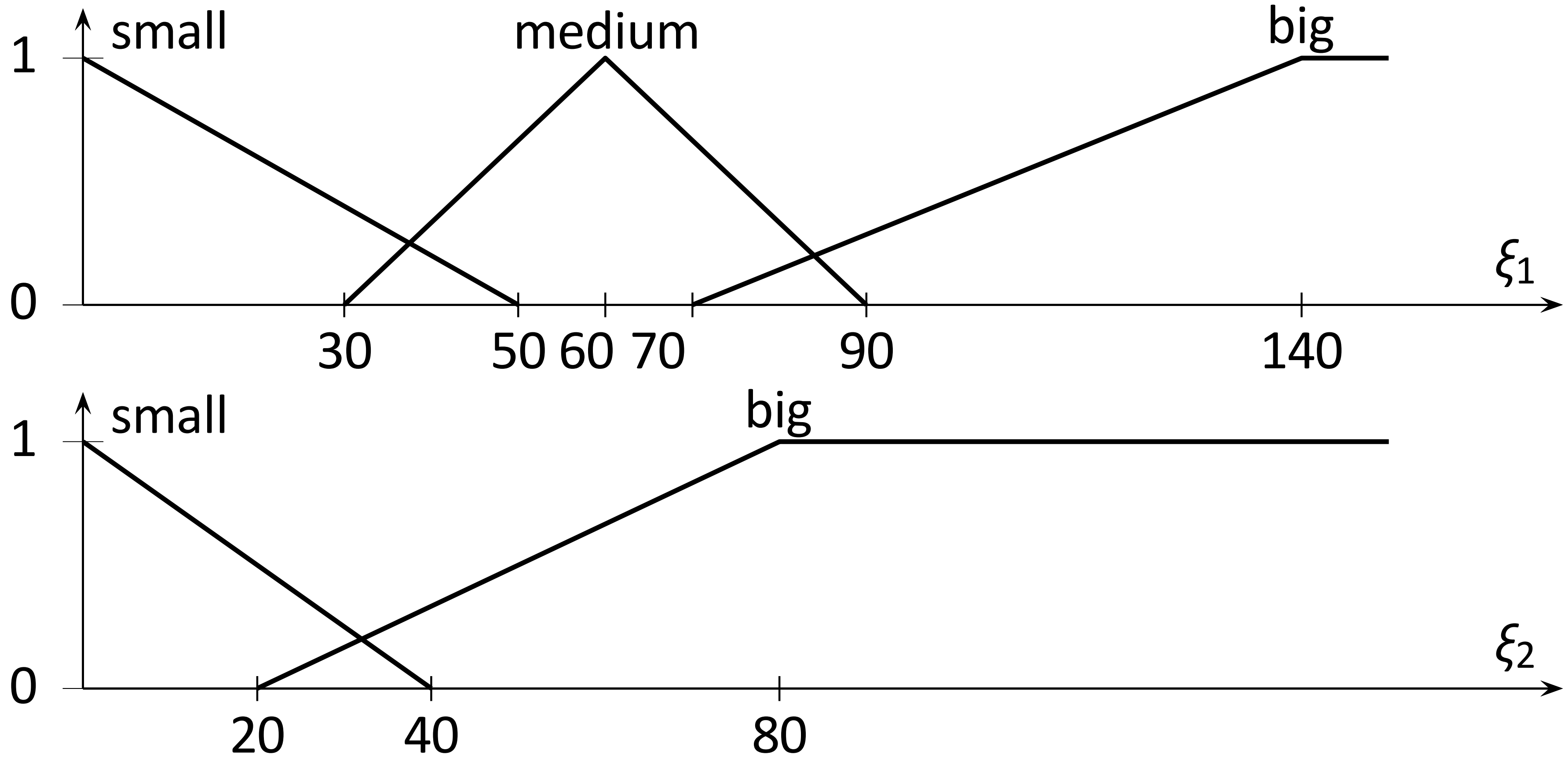
ξ_4 : distance of car to outer barrier

η = rotation speed of steering wheel

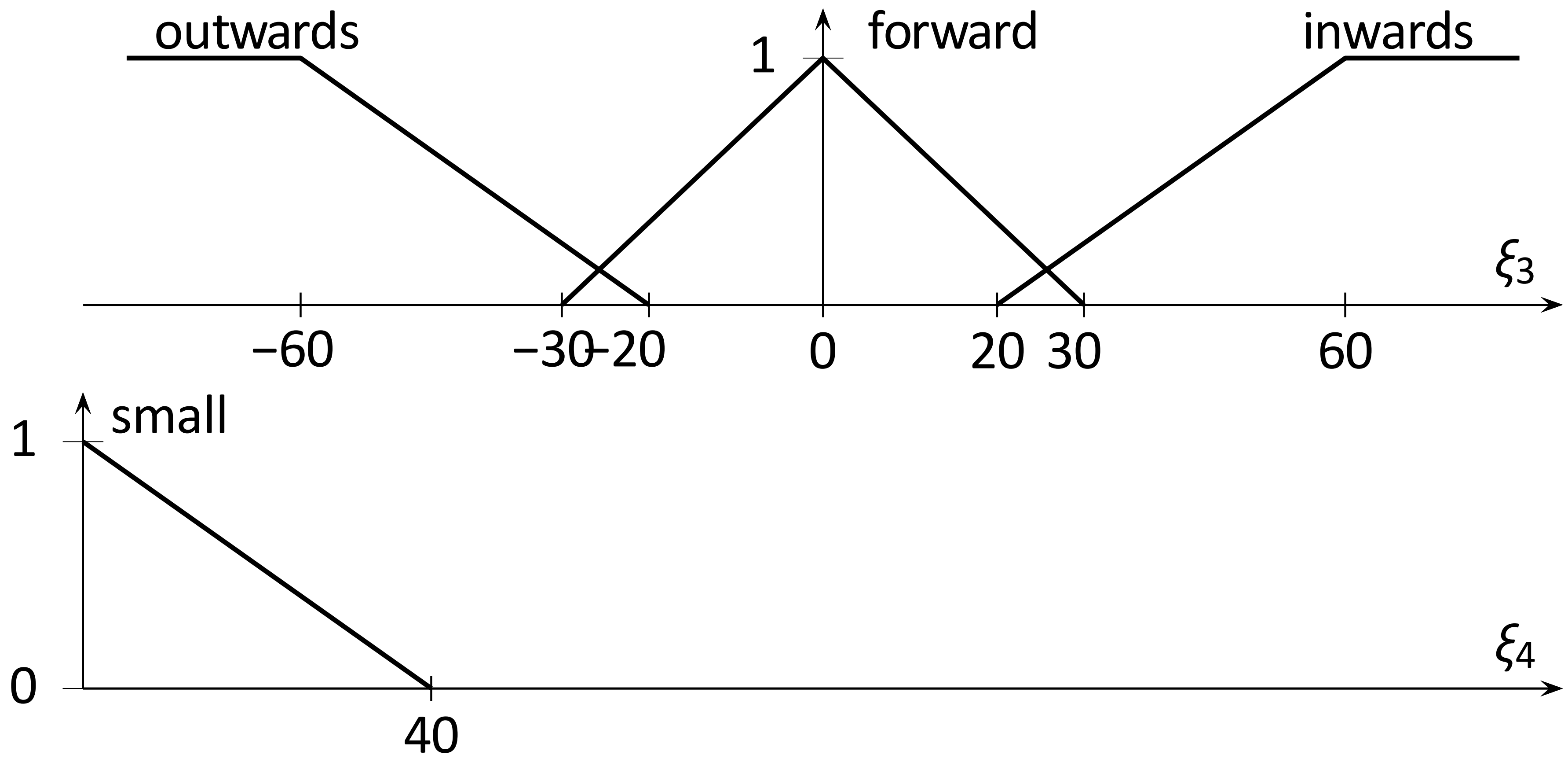
$X_1 = [0\text{cm}, 150\text{cm}]$, $X_2 = [0\text{cm}, 150\text{cm}]$

$X_3 = [-90^\circ, 90^\circ]$, $X_4 = [0\text{cm}, 150\text{cm}]$

Fuzzy Partitions of X_1 and X_2



Fuzzy Partitions of X_3 and X_4



Rules for Car

$$R_r : \textbf{if } \xi_1 \text{ is } A \text{ and } \xi_2 \text{ is } B \text{ and } \xi_3 \text{ is } C \text{ and } \xi_4 \text{ is } D$$

$$\textbf{then } \eta = p_0^{(A,B,C,D)} + p_1^{(A,B,C,D)} \cdot \xi_1 + p_2^{(A,B,C,D)} \cdot \xi_2 + p_3^{(A,B,C,D)} \cdot \xi_3 + p_4^{(A,B,C,D)} \cdot \xi_4$$

$A \in \{small,\ medium,\ big\}$

$B \in \{small,\ big\}$

$C \in \{outwards,\ forward,\ inwards\}$

$D \in \{small\}$

$p_0^{(A,B,C,D)}, \ldots, p_4^{(A,B,C,D)} \in \mathbb{R}$

Control Rules for the Car

rule	ξ_1	ξ_2	ξ_3	ξ_4	p_0	p_1	p_2	p_3	p_4
R_1	-	-	outwards	small	3.000	0.000	0.000	−0.045	−0.004
R_2	-	-	forward	small	3.000	0.000	0.000	−0.030	−0.090
R_3	small	small	outwards	-	3.000	−0.041	0.004	0.000	0.000
R_4	small	small	forward	-	0.303	−0.026	0.061	−0.050	0.000
R_5	small	small	inwards	-	0.000	−0.025	0.070	−0.075	0.000
R_6	small	big	outwards	-	3.000	−0.066	0.000	−0.034	0.000
R_7	small	big	forward	-	2.990	−0.017	0.000	−0.021	0.000
R_8	small	big	inwards	-	1.500	0.025	0.000	−0.050	0.000
R_9	medium	small	outwards	-	3.000	−0.017	0.005	−0.036	0.000
R_{10}	medium	small	forward	-	0.053	−0.038	0.080	−0.034	0.000
R_{11}	medium	small	inwards	-	−1.220	−0.016	0.047	−0.018	0.000
R_{12}	medium	big	outwards	-	3.000	−0.027	0.000	−0.044	0.000
R_{13}	medium	big	forward	-	7.000	−0.049	0.000	−0.041	0.000
R_{14}	medium	big	inwards	-	4.000	−0.025	0.000	−0.100	0.000
R_{15}	big	small	outwards	-	0.370	0.000	0.000	−0.007	0.000
R_{16}	big	small	forward	-	−0.900	0.000	0.034	−0.030	0.000
R_{17}	big	small	inwards	-	−1.500	0.000	0.005	−0.100	0.000
R_{18}	big	big	outwards	-	1.000	0.000	0.000	−0.013	0.000
R_{19}	big	big	forward	-	0.000	0.000	0.000	−0.006	0.000
R_{20}	big	big	inwards	-	0.000	0.000	0.000	−0.010	0.000

Sample Calculation

Assume that the car is 10 cm away from beginning of bend ($\xi_1 = 10$).

The distance of the car to the inner barrier be 30 cm ($\xi_2 = 30$).

The distance of the car to the outer barrier be 50 cm ($\xi_4 = 50$).

The direction of the car be "forward" ($\xi_3 = 0$).

Then according to all rules $R_1, \ldots, R_{20}$,
only premises of R_4 and R_7 have a value $\neq 0$.

Membership Degrees to Control Car

	small	medium	big
$\xi_1 = 10$	0.8	0	0

	small	big
$\xi_2 = 30$	0.25	0.167

	outwards	forward	inwards
$\xi_3 = 0$	0	1	0

	small
$\xi_4 = 50$	0

Sample Calculation (cont.)

For the premise of R_4 and R_7, $\alpha_4 = 1/4$ and $\alpha_7 = 1/6$, resp.

The rules weights $\alpha_4 = \frac{1/4}{1/4+1/6} = 3/5$ for R_4 and $\alpha_5 = 2/5$ for R_7.

R_4 yields

$$\begin{aligned}\eta_4 &= 0.303 - 0.026 \cdot 10 + 0.061 \cdot 30 - 0.050 \cdot 0 + 0.000 \cdot 50 \\ &= 1.873.\end{aligned}$$

R_7 yields

$$\begin{aligned}\eta_7 &= 2.990 - 0.017 \cdot 10 + 0.000 \cdot 30 - 0.021 \cdot 0 + 0.000 \cdot 50 \\ &= 2.820.\end{aligned}$$

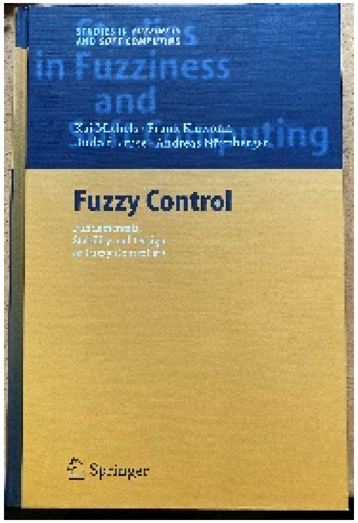
The final value for control variable is thus

$$\eta = 3/5 \cdot 1.873 + 2/5 \cdot 2.820 = 2.2518.$$

Fuzzy Control

Biggest success of fuzzy systems in industry and commerce.

Special kind of model-based non-linear control method.



Examples: technical systems

- Electrical engine moving an elevator,
- Heating installation

Goal: define certain behavior

- Engine should maintain certain number of revolutions per minute.
- Heating should guarantee certain room temperature.