OTTO VON GUERICKE UNIVERSITÄT MAGDEBURG

INF FAKULTÄT FÜR INFORMATIK

Fuzzy Set Operators

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In set theory, **operators** are defined by **propositional logics operator**

Let X be universal set (often called universe of discourse). Then we define

$$A \cap B = \{x \in X \mid x \in A \wedge x \in B\}$$

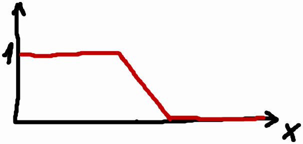
$$A \cup B = \{x \in X \mid x \in A \vee x \in B\}$$

$$A^c = \{x \in X \mid x \notin A\} = \{x \in X \mid \neg (x \in A)\}$$

$A \subseteq B$ if and only if $(x \in A) \rightarrow (x \in B)$ for all $x \in X$

Fuzzy Set Operators can be defined by using **multivalues logics operators**

Standard Fuzzy Set Operators



$(\mu \wedge \mu')(x) := \min\{\mu(x), \mu'(x)\}$ intersection (“AND”),

$(\mu \vee \mu')(x) := \max\{\mu(x), \mu'(x)\}$ union (“OR”),

$\neg\mu(x) := 1 - \mu(x)$ complement (“NOT”).

μ is subset of μ' if and only if $\mu \leq \mu'$.

Theorem
$(\mathrm{F}(X), \wedge, \vee, \neg)$ *is a complete distributive lattice, but no Boolean algebra.*

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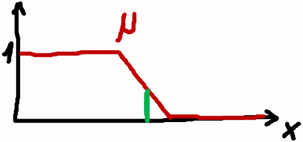

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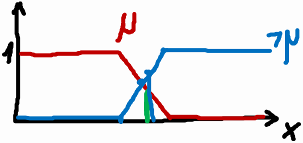

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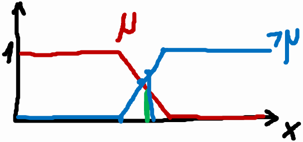

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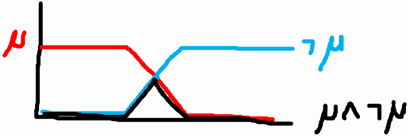


Fuzzy Set Complement

$\bar{\mu}(x) = 1 - \mu(x)$

a	$\bar{a} = 1 - a$
0	1
0.3	0.7
1	0

$\sim$

$N : [0,1] \to [0,1]$

Fuzzy Complement/Fuzzy Negation

Definition

Let X be a given set and $\mu \in \mathcal{F}(X)$. Then the *complement* $\bar{\mu}$ can be defined pointwise by $\bar{\mu}(x) := \sim(\mu(x))$ where $\sim\ : [0,1] \to [0,1]$ satisfies the conditions

$$\sim(0) = 1, \quad \sim(1) = 0$$

and

$$\text{for } x, y \in [0,1],\ x \leq y \implies \sim x \geq \sim y \quad (\sim \text{ is non-increasing}).$$

Abbreviation: $\sim x := \sim(x)$

Strict and Strong Negations

Additional properties may be required

- $x, y \in [0, 1], \; x < y \Longrightarrow \sim x > \sim y$ ($\sim$ is strictly decreasing)
- $\sim$ is continuous
- $\sim\sim x = x$ for all $x \in [0, 1]$ ($\sim$ is involutive)

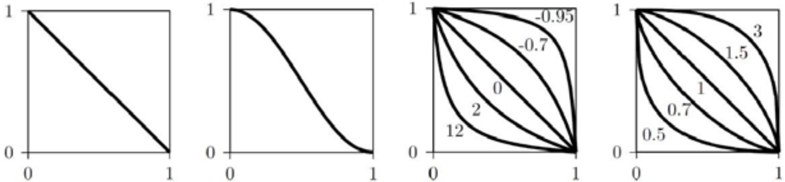
According to conditions, two subclasses of negations are defined:

Definition
A negation is called *strict* if it is also strictly decreasing and continuous. A strict negation is said to be *strong* if it is involutive,too.

$\sim x = 1 - x^2$, for instance, is strict, not strong, thus not involutive

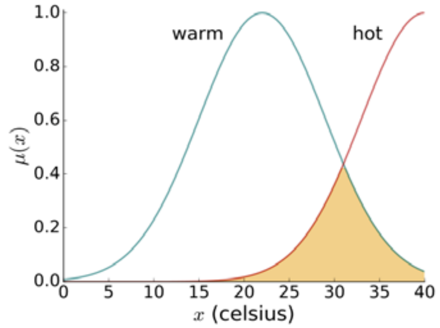
Families of Negations

standard negation:	$\sim x = 1 - x$
threshold negation:	$\sim_\theta(x) = \begin{cases} 1 & \text{if } x \le \theta \\ 0 & \text{otherwise} \end{cases}$
Cosine negation:	$\sim x = \frac{1}{2}(1 + \cos(\pi x))$
Sugeno negation:	$\sim_\lambda(x) = \frac{1 - x}{1 + \lambda x}, \quad \lambda > -1$
Yager negation:	$\sim_\lambda(x) = (1 - x^\lambda)^{\frac{1}{\lambda}}$

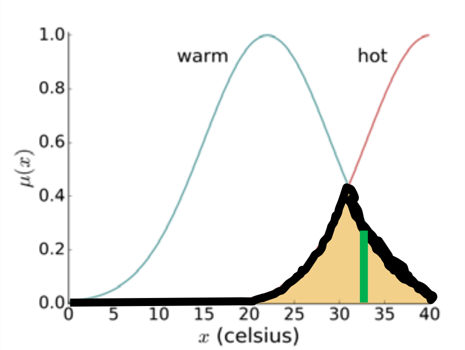


standard cosine Sugeno Yager

Fuzzy Set Intersection and Union



warm and hot ?



Zadeh' Intersection

a and b = min (a,b), for all membership degrees a,b

$(\mu_{warm} \cap \mu_{hot})(x) = \min(\mu_{warm}(x), \mu_{hot}(x))$, for all real numbers x

Classical Intersection and Union

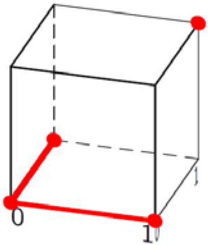
Classical set intersection represents logical conjunction.

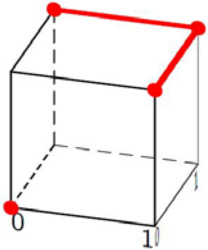
Classical set union represents logical disjunction.

Generalization from $\{0, 1\}$ to $[0, 1]$ as follows:

$x \wedge y$	0	1
0	0	0
1	0	1

$x \vee y$	0	1
0	0	1
1	1	1





Fuzzy Set Intersection and Union

Let A, B be fuzzy subsets of X, *i.e.* $A, B \in \mathcal{F}(X)$.

Their **intersection** and **union** are often defined pointwise using:

$(A \cap B)(x) = \top(A(x), B(x))$ where $\top : [0, 1]^2 \to [0, 1]$

$(A \cup B)(x) = \bot(A(x), B(x))$ where $\bot : [0, 1]^2 \to [0, 1]$.

Triangular Norms and Conorms

$\top$ is a *triangular norm (t-norm)* $\Longleftrightarrow \top$ satisfies conditions T1-T4

$\bot$ is a *triangular conorm (t-conorm)* $\Longleftrightarrow \bot$ satisfies C1-C4

Identity Law

T1: $\top(x, 1) = x$

C1: $\bot(x, 0) = x$

Commutativity

T2: $\top(x, y) = \top(y, x)$

C2: $\bot(x, y) = \bot(y, x)$

Associativity

T3: $\top(x, \top(y, z)) = \top(\top(x, y), z)$

C3: $\bot(x, \bot(y, z)) = \bot(\bot(x, y), z)$

Monotonicity

T4: $y \leq z$ implies $\top(x, y) \leq \top(x, z)$

C4: $y \leq z$ implies $\bot(x, y) \leq \bot(x, z)$.

Triangular Norms and Conorms II

Both identity law and monotonicity respectively imply
$\forall x \in [0,1] : \top(0,x) = 0,$
$\forall x \in [0,1] : \bot(1,x) = 1,$

For any t-norm $\top$: $\top(x,y) \leq \min(x,y)$, for any t-conorm $\bot$: $\bot(x,y) \geq \max(x,y)$.

$x = 1 \Rightarrow T(0, 1) = 0$ and
$x \leq 1 \Rightarrow T(x,0) \leq T(1,0) = T(0,1) = 0$

De Morgan Triplet I

For every $\top$ and strong negation $\sim$, one can define *t*-conorm $\bot$ by

$$\bot(x,y) = \sim \top(\sim x, \sim y), \qquad x,y \in [0,1].$$

Additionally, in this case $\top(x,y) = \sim \bot(\sim x, \sim y)$, $x,y \in [0,1]$.

De Morgan Triplet II

Definition

The triplet $(\top, \bot, \sim)$ is called *De Morgan triplet* if and only if

$\top$ is t-norm, $\bot$ is t-conorm, $\sim$ is strong negation,

$\top, \bot$ and $\sim$ satisfy $\bot(x, y) = \sim \top(\sim x, \sim y)$.

In the following, some important De Morgan triplets will be shown, only the most frequently used and important ones.

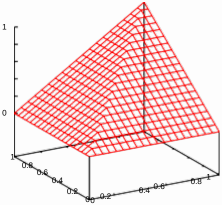
In all cases, the standard negation $\sim x = 1 - x$ is considered.

The Minimum and Maximum I

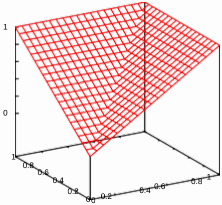
$\top_{\min}(x, y) = \min(x, y), \qquad \bot_{\max}(x, y) = \max(x, y)$

Minimum is the greatest *t*-norm and max is the weakest *t*-conorm.

$\top(x, y) \leq \min(x, y)$ and $\bot(x, y) \geq \max(x, y)$ for any $\top$ and $\bot$



$\top_{\min}$



$\bot_{\max}$

The Special Role of Minimum and Maximum I

$\top_{min}$ and $\bot_{max}$ play key role for intersection and union, resp. In a practical sense, they are very simple.

Apart from the identity law, commutativity, associativity and monotonicity, they also satisfy the following properties for all $x, y, z \in [0, 1]$:

Distributivity

$\bot_{max}(x, \top_{min}(y, z)) = \top_{min}(\bot_{max}(x, y), \bot_{max}(x, z)),$

$\top_{min}(x, \bot_{max}(y, z)) = \bot_{max}(\top_{min}(x, y), \top_{min}(x, z))$

Continuity

$\top_{min}$ and $\bot_{max}$ are continuous.

The Special Role of Minimum and Maximum II

Strict monotonicity on the diagonal

$x < y$ implies $\top_{\min}(x, x) < \top_{\min}(y, y)$ and $\bot_{\max}(x, x) < \bot_{\max}(y, y)$.

Idempotency

$\top_{\min}(x, x) = x, \quad \bot_{\max}(x, x) = x$

Absorption

$\top_{\min}(x, \bot_{\max}(x, y)) = x, \quad \bot_{\max}(x, \top_{\min}(x, y)) = x$

Non-compensation

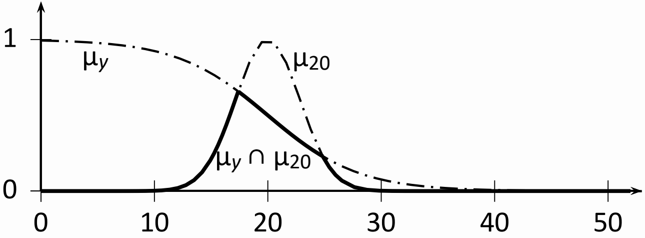
$x < y < z$ imply $\top_{\min}(x, z) \neq \top_{\min}(y, y)$ and $\bot_{\max}(x, z) \neq \bot_{\max}(y, y)$.

The Minimum and Maximum II

$\top_{min}$ and $\bot_{max}$ can be easily processed numerically and visually,

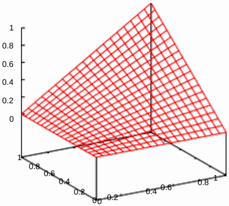
e.g. linguistic values *young* and *approx. 20* described by μ_y, μ_{20}.

$\top_{min}(\mu_y, \mu_{20})$ is shownbelow.

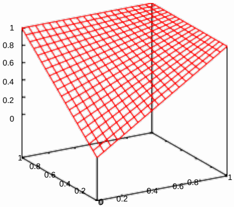


The Product and Probabilistic Sum

$\top_{prod}(x, y) = x \cdot y,$ $\qquad \bot_{sum}(x, y) = x + y - x \cdot y$



$\top_{prod}$

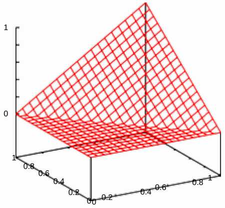


$\bot_{sum}$

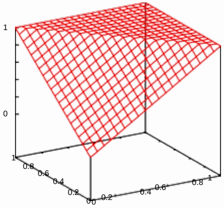
The Łukasiewicz t-norm and t-conorm

$\top_{\text{Łuka}}(x, y) = \max\{0,\ x + y - 1\}$, $\bot_{\text{Łuka}}(x, y) = \min\{1,\ x + y\}$

$\top_{\text{Łuka}}$, $\bot_{\text{Łuka}}$ are also called *bold intersection* and *boundedsum*.



$\top_{\text{Łuka}}$



$\bot_{\text{Łuka}}$

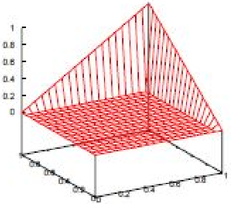
The Drastic Product and Sum

$$\top_{-1}(x,y) = \begin{cases} \min(x,y) & \text{if } \max(x,y) = 1 \\ 0 & \text{otherwise} \end{cases}$$

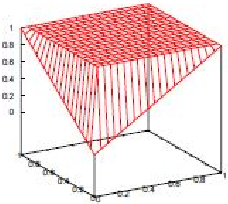
$$\bot_{-1}(x,y) = \begin{cases} \max(x,y) & \text{if } \min(x,y) = 0 \\ 1 & \text{otherwise} \end{cases}$$

$\top_{-1}$ is the weakest t-norm, $\bot_{-1}$ is the strongest t-conorm.

$\top_{-1} \leq \top \leq \top_{\min}, \quad \bot_{\max} \leq \bot \leq \bot_{-1}$ for any $\top$ and $\bot$

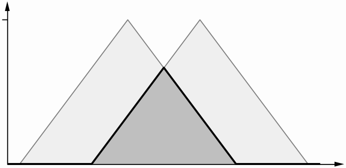


$\top_{-1}$

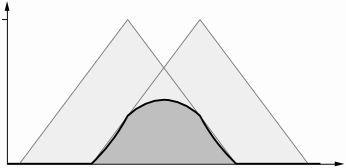


$\bot_{-1}$

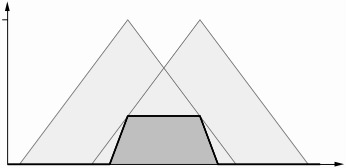
Examples of Fuzzy Intersections



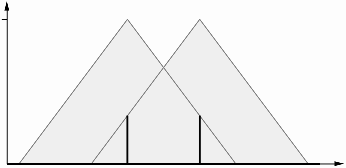
t-norm T_{min}



t-norm T_{prod}



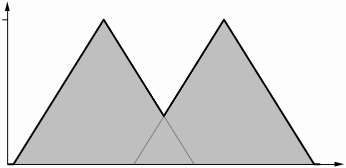
t-norm $T_{Łuka}$



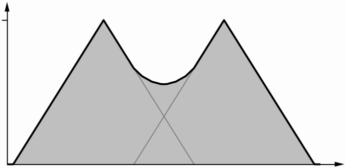
t-norm T_{-1}

Note that all fuzzy intersections are contained within upper left graph and lower right one.

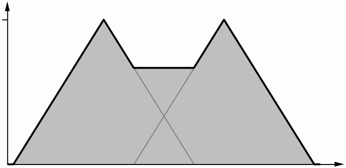
Examples of Fuzzy Unions



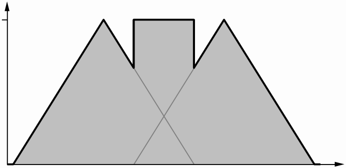
t-conorm $\bot_{\max}$



t-conorm $\bot_{\text{sum}}$



t-conorm $\bot_{\text{Łuka}}$



t-conorm $\bot_{-1}$

Note that all fuzzy unions are contained within upper left graph and lower right one.

Continuous Archimedian *t*-norms and *t*-conorms

Often it is possible to representation functions with several inputs by a function with only one input , *e.g.*

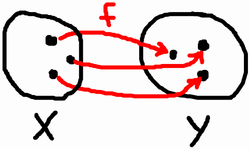
$$K(x,y) = f^{(-1)}(f(x) + f(y))$$

For a subclass of *t*-norms this is possible. The trick makes calculations simpler.

A *t*-norm $\top$ is called
(a) *continuous* if $\top$ is continuous
(b) *Archimedian* if $\top$ is continuous and $\top(x,x) < x$ for all $x \in]0,1[$.

A *t*-conorm $\bot$ is called
(a) *continuous* if $\bot$ is continuous,

(b) *Archimedian* if $\bot$ is continuous and $\bot(x,x) > x$ for all $x \in]0,1[$.

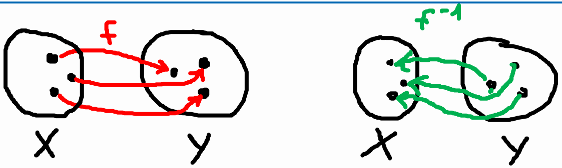


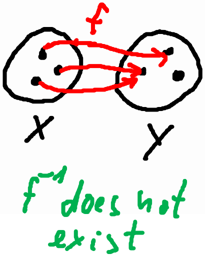
The concept of a pseudoinverse

Definition

Let $f : [a, b] \to [c, d]$ be a monotone function between two closed subintervals of extended real line. The pseudoinverse function to f is the function $f^{(-1)} : [c, d] \to [a, b]$ defined as

$$f^{(-1)}(y) = \begin{cases} \sup\{x \in [a, b] \mid f(x) < y\} & \text{for } f \text{ non-decreasing,} \\ \sup\{x \in [a, b] \mid f(x) > y\} & \text{for } f \text{ non-increasing.} \end{cases}$$





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Archimedian t-norms

Theorem
A t-norm T *is Archimedian if and only if there exists a strictly decreasing and continuous function* $f : [0,1] \to [0,\infty)$ *with* $f(1) = 0$ *such that*

$$\top(x,y) = f^{(-1)}(f(x) + f(y)) \tag{1}$$

where

$$f^{(-1)}(x) = \begin{cases} f^{-1}(x) & \text{if } x \le f(0) \\ 0 & \text{otherwise} \end{cases}$$

is the pseudoinverse of f. *Moreover, this representation is unique up to a positive multiplicative constant.*

$\top$ *is generated by* f if $\top$ has representation (1).

f is called *additive generator* of $\top$.

Additive Generators of t-norms – Examples

Find an additive generator f of $\top_{\text{Łuka}}(x, y) = \max\{x + y - 1,\ 0\}$.

for instance $f_{\text{Łuka}}(x) = 1 - x$

then, $f_{\text{Łuka}}^{(-1)}(x) = \max\{1 - x,\ 0\}$

thus $\top_{\text{Łuka}}(x, y) = f_{\text{Łuka}}^{(-1)}(f_{\text{Łuka}}(x) + f_{\text{Łuka}}(y))$

Find an additive generator f of $\top_{\text{prod}}(x, y) = x \cdot y$.

to be discussed in the exercise

hint: use of logarithmic and exponential function

Archimedean t-conorms

Theorem

A t-conorm $\bot$ *Archimedean if and only if there exists a strictly increasing and continuous function* $g : [0,1] \to [0,\infty]$ *with* $g(0) = 0$ *such that*

$$\bot(x,y) = g^{(-1)}(g(x) + g(y)) \qquad (2)$$

where

$$g^{(-1)}(x) = \begin{cases} g^{-1}(x) & \text{if } x \leq g(1) \\ 1 & \text{otherwise} \end{cases}$$

is the pseudoinverse of g. *Moreover, this representation is unique up to a positive multiplicative constant.*

$\bot$ *is generated by* g if $\bot$ has representation (2).

g is called *additive generator* of $\bot$.

Additive Generators of t-conorms – Two Examples

Find an additive generator g of $\bot_{\text{Łuka}}(x, y) = \min\{x + y,\ 1\}$.

for instance $g_{\text{Łuka}}(x) = x$

then, $g_{\text{Łuka}}^{(-1)}(x) = \min\{x,\ 1\}$

thus $\bot_{\text{Łuka}}(x, y) = g_{\text{Łuka}}^{(-1)}(g_{\text{Łuka}}(x) + g_{\text{Łuka}}(y))$

Find an additive generator g of $\bot_{\text{sum}}(x, y) = x + y - x \cdot y$.

to be discussed in the exercise

hint: use of logarithmic and exponential function

Now, let us examine some typical families of operations.

Sugeno-Weber Family I

For $\lambda > -1$ and $x, y \in [0, 1]$, define

$$\top_\lambda(x, y) = \max\left\{\frac{x + y - 1 + \lambda xy}{1 + \lambda},\ 0\right\},$$

$$\bot_\lambda(x, y) = \min\{x + y + \lambda xy,\ 1\}.$$

$\lambda = 0$ leads to $\top_{\text{Łuka}}$ and $\bot_{\text{Łuka}}$, resp.

$\lambda \to \infty$ results in $\top_{\text{prod}}$ and $\bot_{\text{sum}}$, resp.

$\lambda \to -1$ creates $\top_{-1}$ and $\bot_{-1}$, resp.

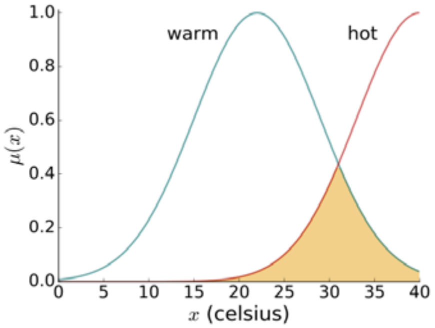
Sugeno-Weber Family II

Additive generators f_λ of $\top_\lambda$ are

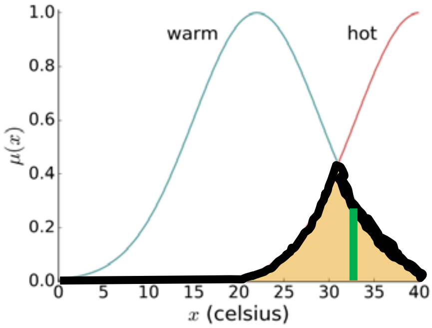
$$f_\lambda(x) = \begin{cases} 1 - x & \text{if } \lambda = 0 \\ 1 - \frac{\log(1+\lambda x)}{\log(1+\lambda)} & \text{otherwise.} \end{cases}$$

$\{\top_\lambda\}_{\lambda > -1}$ are increasing functions of parameter λ.

Additive generators of $\bot_\lambda$ are $g_\lambda(x) = 1 - f_\lambda(x)$.



warm and hot ?



Zadeh' Intersection

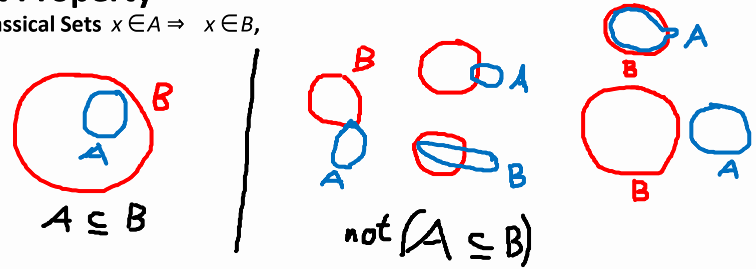
a and b = min (a,b), for all membership degrees a,b

$(\boldsymbol{\mu}_{\text{warm}} \cap \boldsymbol{\mu}_{\text{hot}})(x) = \min(\boldsymbol{\mu}_{\text{warm}}(x), \boldsymbol{\mu}_{\text{hot}}(\mathbf{x}))$, for all real numbers x

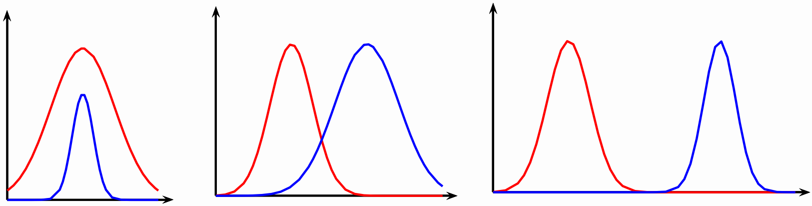
Fuzzy Sets Inclusion

Subset Property

For Classical Sets $x \in A \Rightarrow \quad x \in B,$



For Fuzzy Sets : $x \in \mu \Rightarrow \quad x \in \mu'$



Fuzzy Set Implication

How to model

if speed is fast then distance is high

A straightforward solution with a multivalued logic

- Define fuzzy sets for fast and high
- Determine for all speed values x and all distance values y the membership degrees (i.e. its truth value)
- Calculate for each pair x and y the truth value of the implication

$$\mu_{fast}(x) \Rightarrow \mu_{high}(y)$$

Definition of a Multivalued Implication

1. One way of defining I is to use the property that in classical logic the propositions a ⇒ b and $\neg a \vee b$ have the same truth values for all truth assignments to a and b.
   If we model the disjunction and negation as *t*-conorm and fuzzy complement, resp., then for all $a, b \in [0,1]$ the following defininion of a fuzzy implication seems reasonable:

$$I(a,b) = \bot(\sim a, b).$$

2. Another way is to use the concept of a residuum in classical logic: a ⇒ b and $\max\{x \in \{0,1\} \mid a \wedge x \leq b\}$ have the same truth values for all truth assignments for a, and b. If in a generalized logic the conjunction is modelled by a *t*-norm, then a reasonable generalization could be:

$$I(a,b) = \sup\{x \in [0,1] \mid \top(a,x) \leq b\}.$$

Definition of a Multivalued Implication

3. Another proposal is to use the fact that, in classical logic, the propositions a ⇒ b and $\neg a \vee (a \wedge b)$ have the same truth for all truth assignments.

   A possible extension to many valued logics is therefore

$$I(a, b) = \bot(\sim a, \top(a, b)),$$

   where $(\top, \bot, \sim)$ should be a *De Morgan triplet*.

So again, the classical definition of an implication is unique, whereas there is a „zoo" of fuzzy implications.

Typical question for applications: What to use when and why?

S-Implications

Implications based on $I(a, b) = \bot(\sim a, b)$ are called S-**implications**.

Symbol S is often used to denote t-conorms.

Four well-known S-implications are based on $\sim a = 1 - a$:

Name	$I(a, b)$	$\bot(a, b)$
Kleene-Dienes	$I_{\max}(a, b) = \max(1 - a, b)$	$\max(a, b)$
Reichenbach	$I_{\text{sum}}(a, b) = 1 - a + ab$	$a + b - ab$
Łukasiewicz	$I_{Ł}(a, b) = \min(1,\ 1 - a + b)$	$\min(1,\ a + b)$
largest	$I_{-1}(a, b) = \begin{cases} b, & \text{if } a = 1 \\ 1 - a, & \text{if } b = 0 \\ 1, & \text{otherwise} \end{cases}$	$\begin{cases} b, & \text{if } a = 0 \\ a, & \text{if } b = 0 \\ 1, & \text{otherwise} \end{cases}$

R-Implications

$I(a,b) = \sup\{x \in [0,1] \mid \top(a,x) \le b\}$ leads to R-**implications**.

Symbol R represents close connection to residuated semigroup.

Three well-known R-implications are based on $\sim a = 1 - a$:

- Standard fuzzy intersection leads to **Gödel implication**

$$I_{\min}(a,b) = \sup\{x \mid \min(a,x) \le b\} = \begin{cases} 1, & \text{if } a \le b \\ b, & \text{if } a > b. \end{cases}$$

- Product leads to **Goguen implication**

$$I_{\text{prod}}(a,b) = \sup\{x \mid ax \le b\} = \begin{cases} 1, & \text{if } a \le b \\ b/a, & \text{if } a > b. \end{cases}$$

- Łukasiewicz t-norm leads to **Łukasiewicz implication**

$$I_{\text{Ł}}(a,b) = \sup\{x \mid \max(0, a + x - 1) \le b\} = \min(1,\ 1 - a + b).$$

QL-Implications

Implications based on $I(a,b) = \bot(\sim a, \top(a,b))$ are called **QL-implications** (QL from quantum logic).

Four well-known QL-implications are based on $\sim a = 1 - a$:

- Standard min and max lead to **Zadeh implication**

$$I_Z(a,b) = \max[1-a, \min(a,b)].$$

- The algebraic product and sum lead to

$$I_{\mathrm{p}}(a,b) = 1 - a + a^2 b.$$

- Using $\top_{Ł}$ and $\bot_{Ł}$ leads to **Kleene-Dienes implication** again.
- Using $\top_{-1}$ and $\bot_{-1}$ leads to

$$I_{\mathrm{q}}(a,b) = \begin{cases} b, & \text{if } a = 1 \\ 1 - a, & \text{if } a \neq 1, b \neq 1 \\ 1, & \text{if } a \neq 1, b = 1. \end{cases}$$

All I come from generalizations of the classical implication.

They collapse to the classical implication when truth values are 0 or 1.

Generalizing classical properties leads to following propositions:

1) $a \leq b$ implies $I(a, x) \geq I(b, x)$ *(monotonicity in 1st argument)*
2) $a \leq b$ implies $I(x, a) \leq I(x, b)$ *(monotonicity in 2nd argument)*
3) $I(0, a) = 1$ *(dominance of falsity)*
4) $I(1, b) = b$ *(neutrality of truth)*
5) $I(a, a) = 1$ *(identity)*
6) $I(a, I(b, c)) = I(b, I(a, c))$ *(exchange property)*
7) $I(a, b) = 1$ if and only if $a \leq b$ *(boundary condition)*
8) $I(a, b) = I(\sim b, \sim a)$ for fuzzy complement $\sim$ *(contraposition)*
9) I is a continuous function *(continuity)*

Generator Function

I that satisfy all listed axioms are characterized by this theorem:

Theorem
A function $I : [0,1]^2 \to [0,1]$ *satisfies Axioms 1–9 of fuzzy implications for a particular fuzzy complement* $\sim$ *if and only if there exists a strict increasing continuous function* $f : [0,1] \to [0,\infty)$ *such that* $f(0) = 0$,

$$I(a,b) = f^{(-1)}(f(1) - f(a) + f(b))$$

for all $a, b \in [0,1]$, *and*

$$\sim a = f^{-1}(f(1) - f(a))$$

for all $a \in [0,1]$.

Example

Consider $f_\lambda(a) = \ln(1 + \lambda a)$ with $a \in [0, 1]$ and $\lambda > 0$.

Its pseudo-inverse is

$$f_\lambda^{(-1)}(a) = \begin{cases} \frac{e^a - 1}{\lambda}, & \text{if } 0 \le a \le \ln(1+\lambda) \\ 1, & \text{otherwise.} \end{cases}$$

The fuzzy negation generated by f_λ for all $a \in [0, 1]$ is

$$n_\lambda(a) = \frac{1 - a}{1 + \lambda a}.$$

The resulting fuzzy implication for all $a, b \in [0, 1]$ is thus

$$I_\lambda(a, b) = \min\left(1, \frac{1 - a + b + \lambda b}{1 + \lambda a}\right).$$

If $\lambda \in (-1, 0)$, then I_λ is called **pseudo-Łukasiewicz implication**.