OTTO VON GUERICKE UNIVERSITÄT MAGDEBURG

INF FAKULTÄT FÜR INFORMATIK

Extension Principle

How to “fuzzify” classical methods

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OTTO VON GUERICKE UNIVERSITÄT MAGDEBURG INF FAKULTÄT FÜR INFORMATIK

Addition of Numbers

$+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

$+(3,4) = 3+4 = 7$

$(3,4) \mapsto 7$

Addition of sets

$+ : \mathcal{P}(\mathbb{R}) \times \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$

$(A,B) \mapsto \{x+y \mid x \in A, y \in B\}$

$[1,2] + [3,4] = [4,6]$

How to extend $\phi : X^n \to Y$ to $\hat{\phi} : \mathcal{F}(X)^n \to \mathcal{F}(Y)$?

Let $\mu \in \mathcal{F}(\mathbb{R})$ be a fuzzy set of the imprecise concept "about 2".

Then the degree of membership $\mu(2.2)$ can be seen as *truth value* of the statement "2.2 is about equal to 2".

Let $\mu' \in \mathcal{F}(\mathbb{R})$ be a fuzzy set of the imprecise concept "old".

Then the truth value of "2.2 is about equal 2 **and** 2.2 is old" can be seen as membership degree of 2.2 *w.r.t.* imprecise concept "about 2 and old".

Reminder: Operating with Truth Values

Any $\top$ ($\bot$) can be used to represent conjunction (disjunction).

However, now only $\top_{\min}$ and $\bot_{\max}$ shall be used.

Let P be set of imprecise statements that can be combined by *and, or*.

truth : P $\to$ [0, 1] assigns truth value truth(a) to every $a \in$ P.

truth(a) = 0 means a is definitely false.

truth(a) = 1 means a is definitely true.

If 0 < truth(a) < 1, then only gradual truth of statement a.

Extension Principle

Combination of two statements $a, b \in P$:

$$\begin{aligned} \text{truth}(a \text{ and } b) = \quad & \text{truth}(a \wedge b) = \quad & \min\{\text{truth}(a), \text{truth}(b)\}, \\ \text{truth}(a \text{ or } b) = \quad & \text{truth}(a \vee b) = \quad & \max\{\text{truth}(a), \text{truth}(b)\} \end{aligned}$$

For infinite number of statements $a_i, i \in I$:

$$\begin{aligned} \text{truth}(\forall i \in I : a_i) &= \inf \{\text{truth}(a_i) \mid i \in I\}, \\ \text{truth}(\exists i \in I : a_i) &= \sup \{\text{truth}(a_i) \mid i \in I\} \end{aligned}$$

This concept helps to extend $\phi : X^n \to Y$ to $\hat{\phi} : \mathcal{F}(X)^n \to \mathcal{F}(Y)$.

- Crisp tuple $(x_1, \ldots, x_n)$ is mapped to crisp value $\phi(x_1, \ldots, x_n)$.
- Imprecise descriptions $(\mu_1, \ldots, \mu_n)$ of $(x_1, \ldots, x_n)$ are mapped to fuzzy value $\hat{\phi}(\mu_1, \ldots, \mu_n)$.

Example – How to extend the addition?

$+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \quad (a, b) \mapsto a + b$

Extensions to sets: $+ : 2^{\mathbb{R}} \times 2^{\mathbb{R}} \to 2^{\mathbb{R}}$

$$(A, B) \mapsto A + B = \{y \mid (\exists a)(\exists b)(y = a + b) \wedge (a \in A) \wedge (b \in B)\}$$

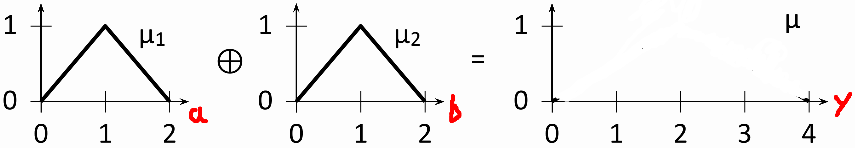
Extensions to fuzzy sets:

$$+ : \mathcal{F}(\mathbb{R}) \times \mathcal{F}(\mathbb{R}) \to \mathcal{F}(\mathbb{R}), \quad (\mu_1, \mu_2) \mapsto \mu_1 \oplus \mu_2$$

$$\begin{aligned} \text{truth}(y \in \mu_1 \oplus \mu_2) &= \text{truth}((\exists a)(\exists b) : (y = a + b) \wedge (a \in \mu_1) \wedge (b \in \mu_2)) \\ &= \sup_{a,b}\{\text{truth}(y = a + b) \wedge \text{truth}(a \in \mu_1) \wedge \\ &\qquad \text{truth}(b \in \mu_2)\} \\ &= \sup_{a,b: y=a+b} \{\min(\mu_1(a),\ \mu_2(b))\} \end{aligned}$$

Example – How to extend the addition?

$$\mu(y) = \sup_{a,b:y=a+b} \{\min(\mu_1(a),\ \mu_2(b))\}$$



$\mu(2) = 1$ because $\mu_1(1) = 1$ and $\mu_2(1) = 1$

$\mu(5) = 0$ because if $a + b = 5$, then $\min\{\mu_1(a), \mu_2(b)\} = 0$

$\mu(1) = 0.5$ because it is the result of an optimization task with optimum at, *e.g.* $a = 0.5$ and $b = 0.5$

Example – How to extend the addition?

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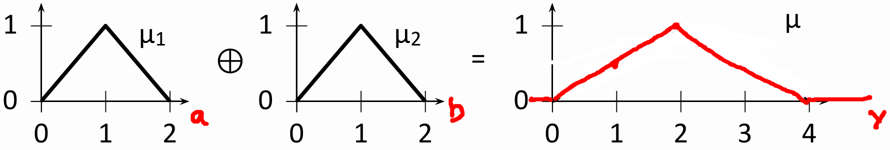

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$\mu(1) = ?$

a	b	$\mu_1(a) \wedge \mu_2(b)$
0	1	$0 \wedge 1 = 0$
1	0	$1 \wedge 0 = 0$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2} \wedge \frac{1}{2} = \frac{1}{2}$
-1	2	$0 \wedge 0 = 0$
1.5	-1.5	$\frac{1}{2} \wedge 0 = 0$
0.2	0.8	$0.2 \wedge 0.8 = 0.2$

Extension to Sets

Definition

Let $\phi : X^n \to Y$ be a mapping. The *extension* $\hat{\phi}$ of ϕ is given by

$$\hat{\phi} : [2^X]^n \to 2^Y \quad \text{with}$$

$$\hat{\phi}(A_1, \ldots, A_n) = \{y \in Y \mid \exists (x_1, \ldots, x_n) \in A_1 \times \ldots \times A_n : \phi(x_1, \ldots, x_n) = y\}.$$

Extension to Fuzzy Sets

Definition

Let $\phi : X^n \to Y$ be a mapping. The *extension* $\hat{\phi}$ of ϕ is given by

$$\hat{\phi} : [\mathcal{F}(X)]^n \to \mathcal{F}(Y) \quad \text{with}$$

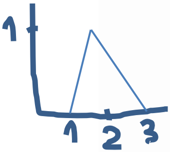
$$\hat{\phi}(\mu_1, \ldots, \mu_n)(y) = \sup\{\min\{\mu_1(x_1), \ldots, \mu_n(x_n)\} \mid (x_1, \ldots, x_n) \in X^n \land \phi(x_1, \ldots, x_n) = y\}$$

assuming that $\sup \emptyset = 0$.

Example I

Let fuzzy set "approximately 2" be defined as

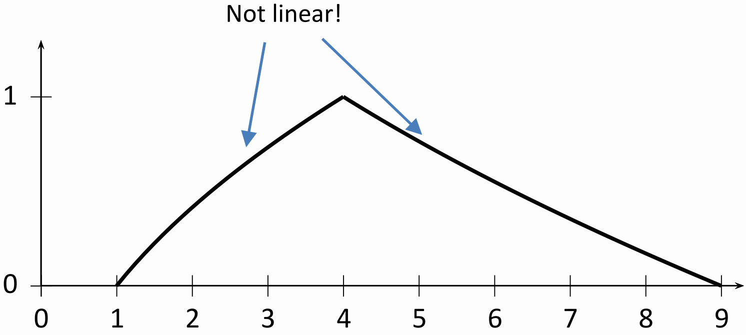
$$\mu(x) = \begin{cases} x-1, & \text{if } 1 \le x \le 2 \\ 3-x, & \text{if } 2 \le x \le 3 \\ 0, & \text{otherwise.} \end{cases}$$



The extension of $\phi : \mathbb{R} \to \mathbb{R}, x \mapsto x^2$ to fuzzy sets on $\mathbb{R}$ is

$$\begin{aligned} \hat{\phi}(\mu)(y) &= \sup\left\{\mu(x) \;\middle|\; x \in \mathbb{R} \wedge x^2 = y\right\} \\ &= \begin{cases} \sqrt{y}-1, & \text{if } 1 \le y \le 4 \\ 3-\sqrt{y}, & \text{if } 4 \le y \le 9 \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Example II



The extension principle is taken as basis for “fuzzifying” whole theories.

Fuzzy Sets of IR

There are many different types of fuzzy sets $\mu : \mathbb{R} \to [0, 1]$,

They play important role in many applications, *e.g.* fuzzy control, decision making, approximate reasoning, optimization, and statistics with imprecise data.

In most applications the fuzzy sets are very simple functions, in some applications these functions are more complicated and difficult to interprete .

Some Special Fuzzy Sets I

Here, we only consider special classes $\mathcal{F}(\mathbb{R})$ of fuzzy sets μ on $\mathbb{R}$.

Definition

(a) $\mathcal{F}_N(\mathbb{R}) \stackrel{\text{def}}{=} \{\mu \in \mathcal{F}(\mathbb{R}) \mid \exists x \in \mathbb{R} : \mu(x) = 1\}$,

(b) $\mathcal{F}_C(\mathbb{R}) \stackrel{\text{def}}{=} \{\mu \in \mathcal{F}_N(\mathbb{R}) \mid \forall \alpha \in (0,1] : [\mu]_\alpha \text{ is compact } \}$,

(c) $\mathcal{F}_I(\mathbb{R}) \stackrel{\text{def}}{=} \{\mu \in \mathcal{F}_N(\mathbb{R}) \mid \forall a, b, c \in \mathbb{R} : c \in [a, b] \Rightarrow$
$\mu(c) \geq \min\{\mu(a), \mu(b)\} \}$.

Some Special Fuzzy Sets II

An element in F_N (IR) is called **normal fuzzy set**:

- It's meaningful if $\mu \in F_N$ (IR) is used as *imprecise description* of an existing (but not precisely defined) set $\subseteq$ IR.
- In such cases it would not be plausible to assign maximum membership degree of 1 to no single real number at all.

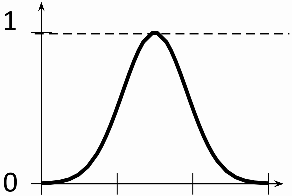
Sets in F_C (IR) are **uppersemi-continuous**:

- Function f is upper semi-continuous at point x_0 iff $\limsup_{x \to x_0} f(x) \leq f(x_0)$.
- This property simplifies arithmetic operations.

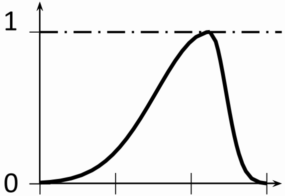
Fuzzy sets in F_I (IR) are called **fuzzy intervals**:

- The are *normal* and *fuzzy convex*.
- Their core is a classical interval.
- If $\mu \in F_I$ (IR) is used for describing an imprecise real number, then often people say: **μ is a fuzzy number.**

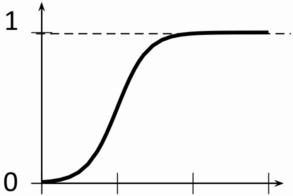
Basic Types of Fuzzy Sets



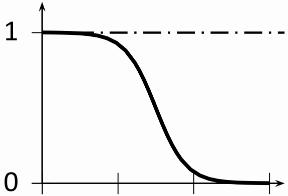
“around *r*”
symmetric bell-shaped



“around *r*”
asymmetric bell-shaped



“large number”
right-open sigmoid



“small number”
left-open sigmoid

Fuzzy Variables

The concept of a fuzzy sets plays fundamental role in formulating *fuzzy variables*.

These are variables whose states are fuzzy sets.

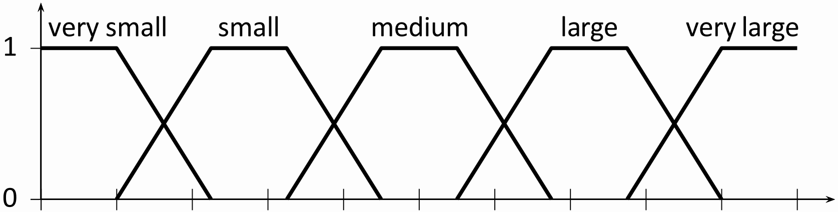
When the fuzzy sets represent linguistic concepts, *e.g. very small*, *small*, *medium*, etc.

then final constructs are called **linguistic variables**.

Each linguistic variable is defined in terms of *base variable* which is a variable in classical sense, *e.g.* temperature, pressure, age.

Linguistic terms representing approximate values of the base variable that are captured by appropriate fuzzy sets.

Linguistic Variables



Each linguistic variable is defined by a quintuple (ν, T, X, g, m), where

- ν *name* of the variable
- T set of *linguistic terms* of ν
- $X \subseteq \mathbb{R}$ *base*
- g grammar: *syntactic rules* for generating a language of linguistic terms
- m meaning: $m : T \rightarrow F(X)$ assigns the semantics $m(t)$ to every $t \in T$

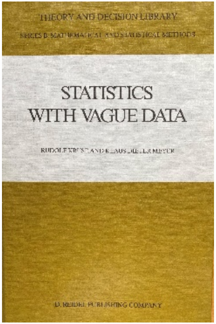
Operations on Linguistic Variables

To deal with linguistic variables, consider

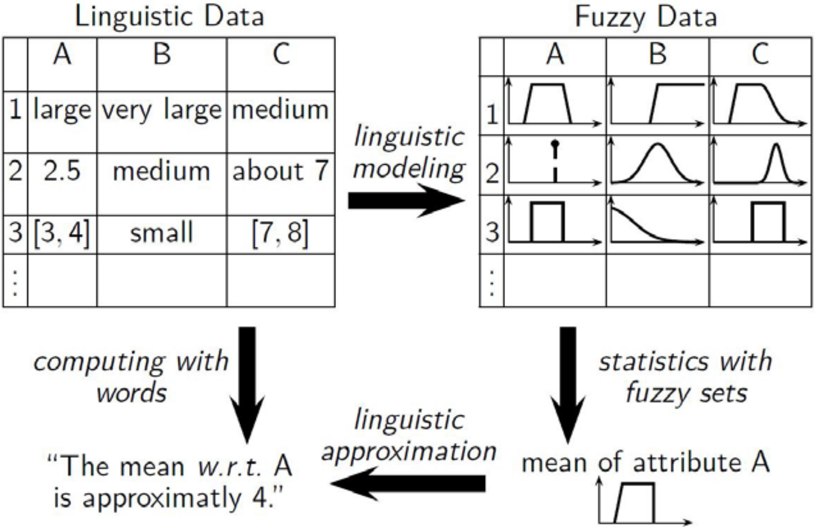
- not only set-theoretic operations
- but also arithmetic operations on fuzzy numbers (*i.e.* interval arithmetic).

Statistics with linguistic data

- Given a sample = (*small*, *medium*, *small*, *large*, ...).
- How to define mean value or standard deviation?



Analysis of Linguistic Data



Example – Application of Linguistic Data

Consider the problem to model the climatic conditions of several towns.

A tourist may want information about tourist attractions.

Assume that linguistic random samples are based on subjective observations of selected people, *e.g.*

- climatic attribute *clouding*
- linguistic values *cloudless, clear, fair, cloudy, …*

Example – Linguistic Modeling by an Expert

The attribute *clouding* is modeled by elementary linguistic values, *e.g*

$$\begin{aligned}
\text{cloudless} &\mapsto \text{sigmoid}(0, -0.07)\\
\text{clear} &\mapsto \text{Gauss}(25, 15)\\
\text{fair} &\mapsto \text{Gauss}(50, 20)\\
\text{cloudy} &\mapsto \text{Gauss}(75, 15)\\
\text{overcast} &\mapsto \text{sigmoid}(100, 0.07)\\
\text{exactly}(x) &\mapsto \text{exact}(x)\\
\text{approx}(x) &\mapsto \text{Gauss}(x, 3)\\
\text{between}(x, y) &\mapsto \text{rectangle}(x, y)\\
\text{approx_between}(x, y) &\mapsto \text{trapezoid}(x - 20, x, y, y + 20)
\end{aligned}$$

where $x, y \in [0, 100] \subseteq \mathbb{R}$.

Example

Gauss(a, b) is, *e.g.* a function defined by

$$f(x) = \exp\left(-\left(\frac{x-a}{b}\right)^2\right), \qquad x, a, b \in \mathbb{R}, \quad b > 0$$

induced language of expressions:

<expression> := <elementary linguistic value> |
(<expression>) |
{ not | dil | con | int } <expression> |
<expression> { and | or } <expression> ,

e.g. approx(x) *and cloudy* is represented by function

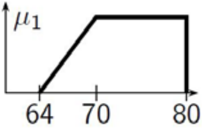
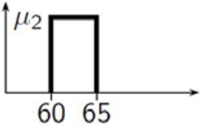
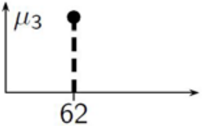
$$\min\{\text{Gauss}(x, 3), \text{Gauss}(75, 15)\}.$$

Example – Linguistic Random Sample

Attribute	:	Clouding
Observations	:	Limassol, Cyprus
2009/10/23	:	cloudy
2009/10/24	:	dil approx_between(50, 70)
2009/10/25	:	fair or cloudy
2009/10/26	:	approx(75)
2009/10/27	:	dil(clear or fair)
2009/10/28	:	int cloudy
2009/10/29	:	con fair
2009/11/30	:	approx(0)
2009/11/31	:	cloudless
2009/11/01	:	cloudless or dil clear
2009/11/02	:	overcast
2009/11/03	:	cloudy and between(70, 80)
. ..	:	. . .
2009/11/10	:	clear

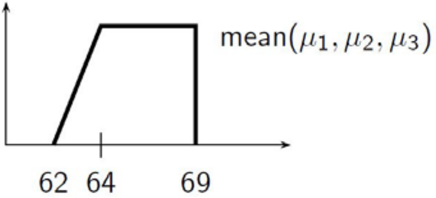
Statistics with fuzzy sets are necessary to analyze linguistic data.

Example – Ling. Random Sample of 3 People

no.	age (linguistic data)	age (fuzzy data)
1	approx. between 70 and 80 and definitely not older than 80	
2	between 60 and 65	
3	62	

Example – Mean Value of Ling. Random Sample

$$\text{mean}(\mu_1, \mu_2, \mu_3) = \frac{1}{3}(\mu_1 \oplus \mu_2 \oplus \mu_3)$$



i.e. approximately between 64 and 69 but not older than 69

Efficient Operations I

How to define arithmetic operations for calculating with $\mathcal{F}(\mathbb{R})$?

Using extension principle for sum $\mu \oplus \mu'$, product $\mu \odot \mu'$ and reciprocal value $\text{rec}(\mu)$ of arbitrary fuzzy sets $\mu, \mu' \in \mathcal{F}(\mathbb{R})$

$$(\mu \oplus \mu')(t) = \sup \{\min\{\mu(x_1), \mu'(x_2)\} \mid x_1, x_2 \in \mathbb{R}, x_1 + x_2 = t\},$$
$$(\mu \odot \mu')(t) = \sup \{\min\{\mu(x_1), \mu'(x_2)\} \mid x_1, x_2 \in \mathbb{R}, x_1 \cdot x_2 = t\},$$
$$\text{rec}(\mu)(t) = \sup \left\{\mu(x) \,\middle|\, x \in \mathbb{R} \setminus \{0\}, \frac{1}{x} = t\right\}.$$

In general, operations on fuzzy sets are much more complicated (especially if vertical instead of horizontal representation is applied).

It's desirable to reduce fuzzy arithmetic to ordinary set arithmetic.

Then, we apply elementary operations of *interval arithmetic*.

Efficient Operations II

Definition
A family $(A_\alpha)_{\alpha\in(0,1)}$ of sets is called *set representation* of $\mu \in \mathcal{F}_N(\mathbb{R})$ if

(a) $0 < \alpha < \beta < 1 \Longrightarrow A_\beta \subseteq A_\alpha \subseteq \mathbb{R}$ and

(b) $\mu(t) = \sup\{\alpha \in [0,1] \mid t \in A_\alpha\}$

holds where $\sup \emptyset = 0$.

Theorem
Let $\mu \in F_N(\mathbb{R})$. The family $(A_\alpha)_{\alpha\in(0,1)}$ of sets is a set representation of μ if and only if

$$[\mu]_{\underline{\alpha}} = \{t \in \mathbb{R} \mid \mu(t) > \alpha\} \subseteq A_\alpha \subseteq \{t \in \mathbb{R} \mid \mu(t) \geq \alpha\} = [\mu]_\alpha$$

is valid for all $\alpha \in (0,1)$.

Efficient Operations III

Theorem

Let $\mu_1, \mu_2, \ldots, \mu_n$ be normal fuzzy sets of $\mathbb{R}$ and $\phi : \mathbb{R}^n \to \mathbb{R}$ be a mapping. Then the following holds:

(a) $\forall \alpha \in [0, 1) : [\hat{\phi}(\mu_1, \ldots, \mu_n)]_{\underline{\alpha}} = \phi([\mu_1]_{\underline{\alpha}}, \ldots, [\mu_n]_{\underline{\alpha}})$,

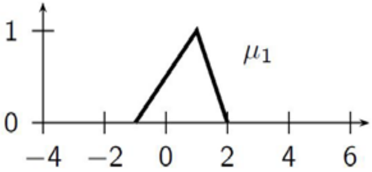
(b) $\forall \alpha \in (0, 1] : [\hat{\phi}(\mu_1, \ldots, \mu_n)]_{\alpha} \supseteq \phi([\mu_1]_{\alpha}, \ldots, [\mu_n]_{\alpha})$,

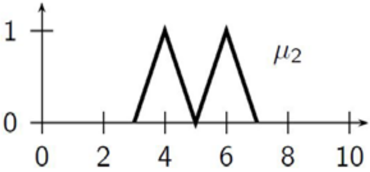
(c) if $((A_i)_\alpha)_{\alpha \in (0,1)}$ *is a set representation of* μ_i *for* $1 \leq i \leq n$, *then* $(\phi((A_1)_\alpha, \ldots, (A_n)_\alpha))_{\alpha \in (0,1)}$ *is a set representation of* $\hat{\phi}(\mu_1, \ldots, \mu_n)$.

For arbitrary mapping ϕ, set representation of its extension $\hat{\phi}$ can be obtained with help of set representation $((A_i)_\alpha)_{\alpha \in (0,1)}, i = 1, 2, \ldots, n$.

It's used to carry out arithmetic operations on fuzzy sets efficiently.

Example I





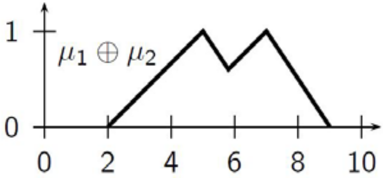
For μ_1, μ_2, the set representations are

- $[\mu_1]_\alpha = [2\alpha - 1, 2 - \alpha]$,
- $[\mu_2]_\alpha = [\alpha + 3, 5 - \alpha] \cup [\alpha + 5, 7 - \alpha]$.

Let $\text{add}(x, y) = x + y$, then $(A_\alpha)_{\alpha \in (0,1)}$ represents $\mu_1 \oplus \mu_2$

$$\begin{aligned} A_\alpha &= \text{add}([\mu_1]_\alpha, [\mu_2]_\alpha) = [3\alpha + 2, 7 - 2\alpha] \cup [3\alpha + 4, 9 - 2\alpha] \\ &= \begin{cases} [3\alpha + 2, 7 - 2\alpha] \cup [3\alpha + 4, 9 - 2\alpha], & \text{if } \alpha \geq 0.6 \\ [3\alpha + 2, 9 - 2\alpha], & \text{if } \alpha \leq 0.6. \end{cases} \end{aligned}$$

Example II



$$(\mu_1 \oplus \mu_2)(x) = \begin{cases} \frac{x-2}{3}, & \text{if } 2 \le x \le 5 \\ \frac{7-x}{2}, & \text{if } 5 \le x \le 5.8 \\ \frac{x-4}{3}, & \text{if } 5.8 \le x \le 7 \\ \frac{9-x}{2}, & \text{if } 7 \le x \le 9 \\ 0, & \text{otherwise} \end{cases}$$

Interval Arithmetic

Determining the set representations of arbitrary combinations of fuzzy sets can be reduced very often to simple interval arithmetic.

Using fundamental operations of arithmetic leads to the following ($a, b, c, d \in \mathbb{R}$):

$$[a, b] + [c, d] = [a + c, b + d]$$
$$[a, b] - [c, d] = [a - d, b - c]$$

$$[a, b] \cdot [c, d] = \begin{cases} [ac, bd], & \text{for } a \geq 0 \wedge c \geq 0 \\ [bd, ac], & \text{for } b < 0 \wedge d < 0 \\ [\min\{ad, bc\}, \max\{ad, bc\}], & \text{for } ab \geq 0 \wedge cd \geq 0 \wedge ac < 0 \\ [\min\{ad, bc\}, \max\{ac, bd\}], & \text{for } ab < 0 \vee cd < 0 \end{cases}$$

$$\frac{1}{[a, b]} = \begin{cases} \left[\frac{1}{b}, \frac{1}{a}\right], & \text{if } 0 \notin [a, b] \\ \left[\frac{1}{b}, \infty\right) \cup \left(-\infty, \frac{1}{a}\right], & \text{if } a < 0 \wedge b > 0 \\ \left[\frac{1}{b}, \infty\right), & \text{if } a = 0 \wedge b > 0 \\ \left(-\infty, \frac{1}{a}\right], & \text{if } a < 0 \wedge b = 0 \end{cases}$$

Interval Arithmetic II

In general, set representation of α-cuts of extensions $\hat{\phi}(\mu_1, \dots, \mu_n)$ cannot be determined directly from α-cuts.

It only works always for continuous ϕ and fuzzy sets in $\mathcal{F}_C(\mathbb{R})$.

Theorem

Let $\mu_1, \mu_2, \dots, \mu_n \in \mathcal{F}_C(\mathbb{R})$ *and* $\phi : \mathbb{R}^n \to \mathbb{R}$ *be a continuous mapping. Then*

$$\forall \alpha \in (0, 1] : [\hat{\phi}(\mu_1, \dots, \mu_n)]_\alpha = \phi([\mu_1]_\alpha, \dots, [\mu_n]_\alpha).$$

So, a horizontal representation is better than a vertical one.
Finding $\hat{\phi}$ values is easier than directly applying the extension principle.
However, all α-cuts cannot be stored in a computer.
Only a finite number of α-cuts can be stored.