Approximate Reasoning

Prof. Dr. Rudolf Kruse

Approximate Reasoning with Fuzzy Rules

Experts like to describe their knowledge by using rules:

Rule 1:	**if** X is M_1, **then** Y is N_1
Rule 2:	**if** X is M_2, **then** Y is N_2
:	:
Rule r :	**if** X is M_r , **then** Y is N_r'

Given *r* **if-then rules** and fact "X is M'", we want to conclude "Y is N'".

This modus ponens style of reasoning is often used, e.g. for PROLOG.

- Often the rules are inherently imprecise/fuzzy.
- What is the semantic of a fuzzy rule, and how to use rule systems in real world applications?

Interpretation 1

Rules are „Patches“

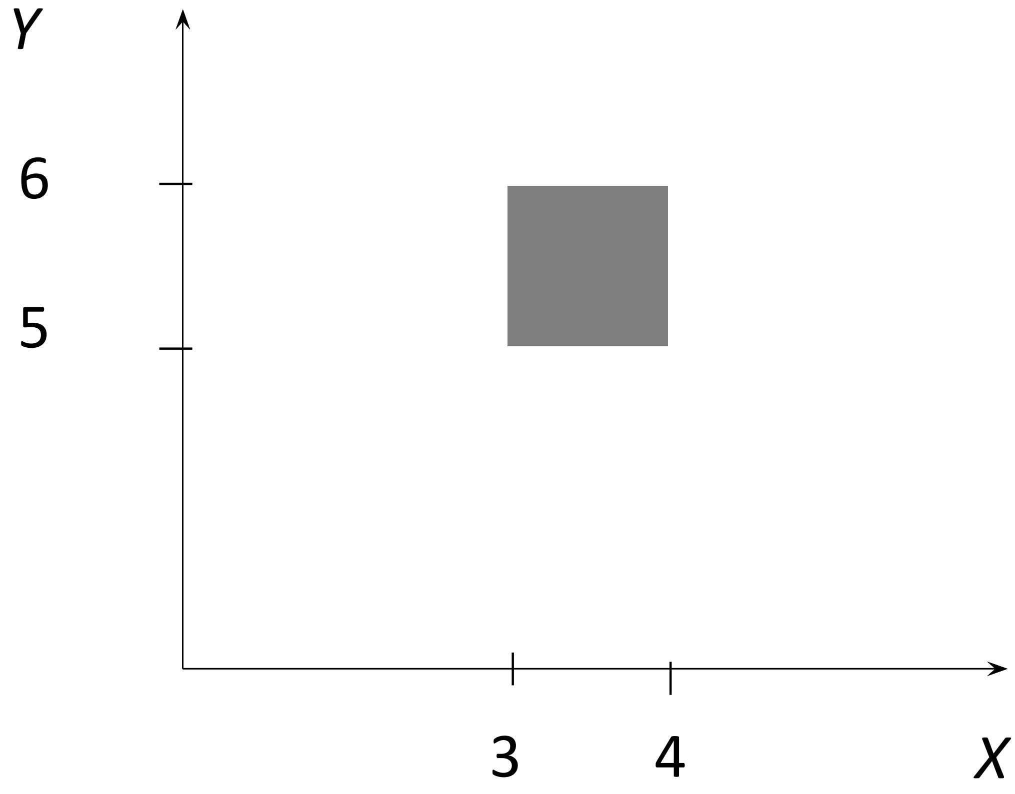
Rules as „Patches“

A rule is interpreted as a patch, where a (control) function passes.

Example

Imprecise rule: **if** $X = [3, 4]$ **then** $Y = [5, 6]$.

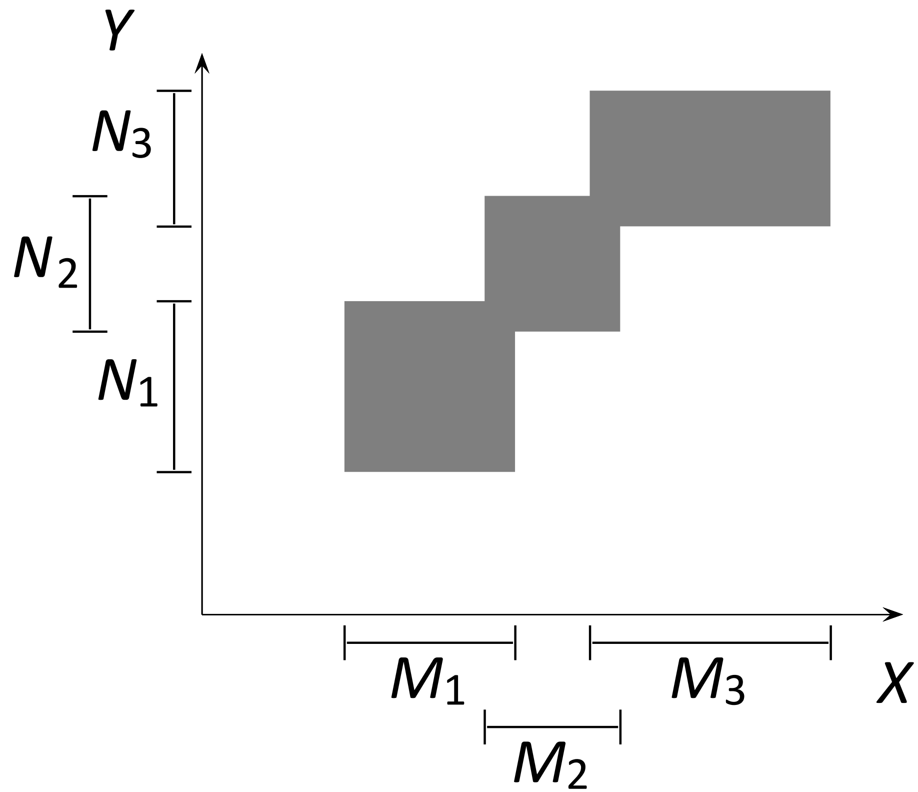
Interpretation : $[3, 4] \times [5, 6]$ is a „patch“, where the function „passes“.



Rules a „Patches“

Several imprecise rules: **if** $X = M_1$ **then** $Y = N_1$, **if** $X = M_2$ **then** $Y = N_2$, **if** $X = M_3$ **then** $Y = N_3$, …

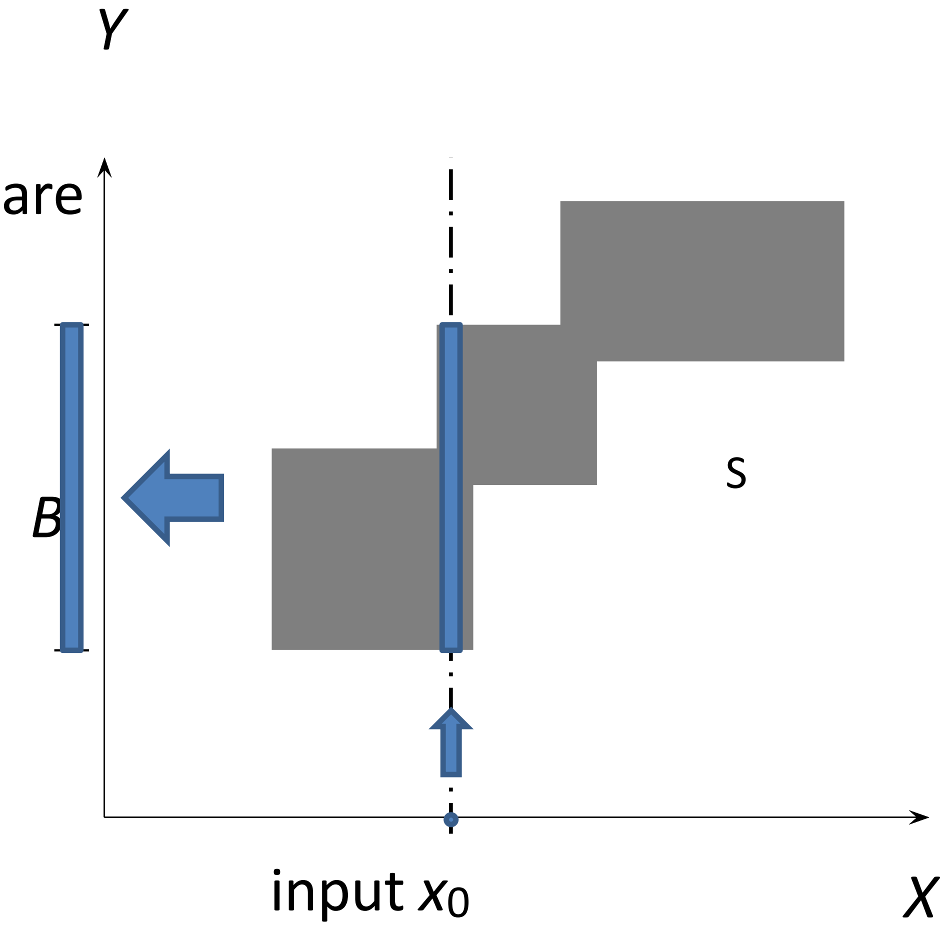
Interpretation 1: Several rules form a “patchwork rug” for the function’s graph. A disjunctive view:



$$S = \bigcup_{i=1}^{r} M_i \times N_i$$

Plausible Output

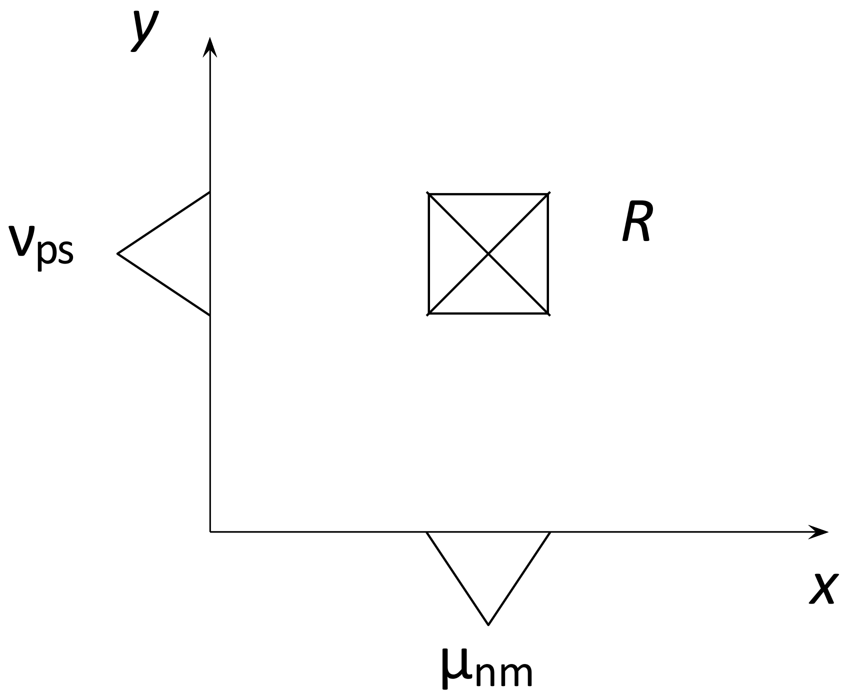
Plausible outputs are in $B = \{x_0\} \circ S$



Fuzzy Rules as Fuzzy Patches

one fuzzy rule:

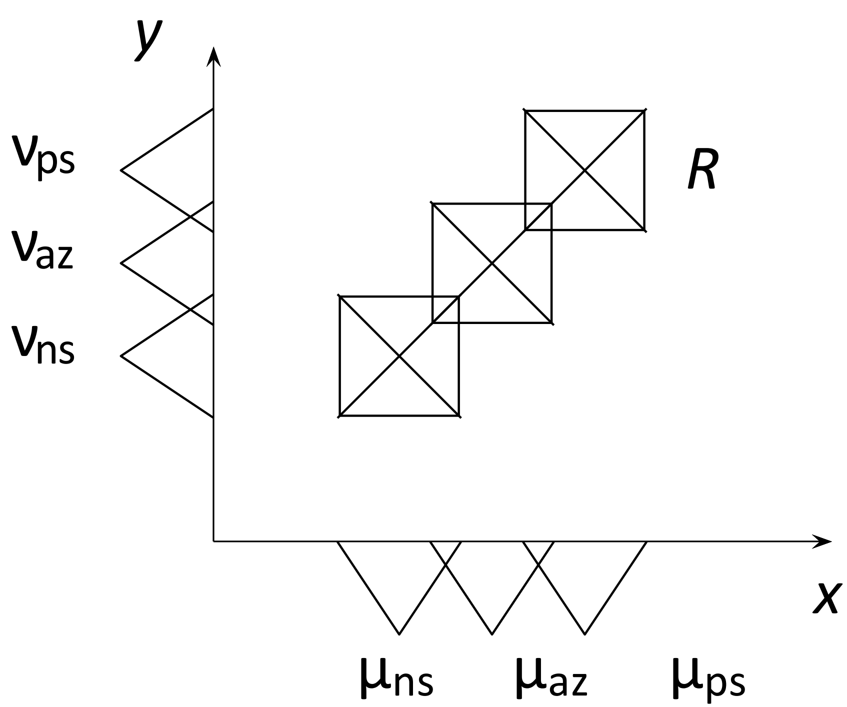
if *X* = nm **then** *Y* = ps



$R = \mu_{nm} \times \nu_{ps}$

several fuzzy rules:

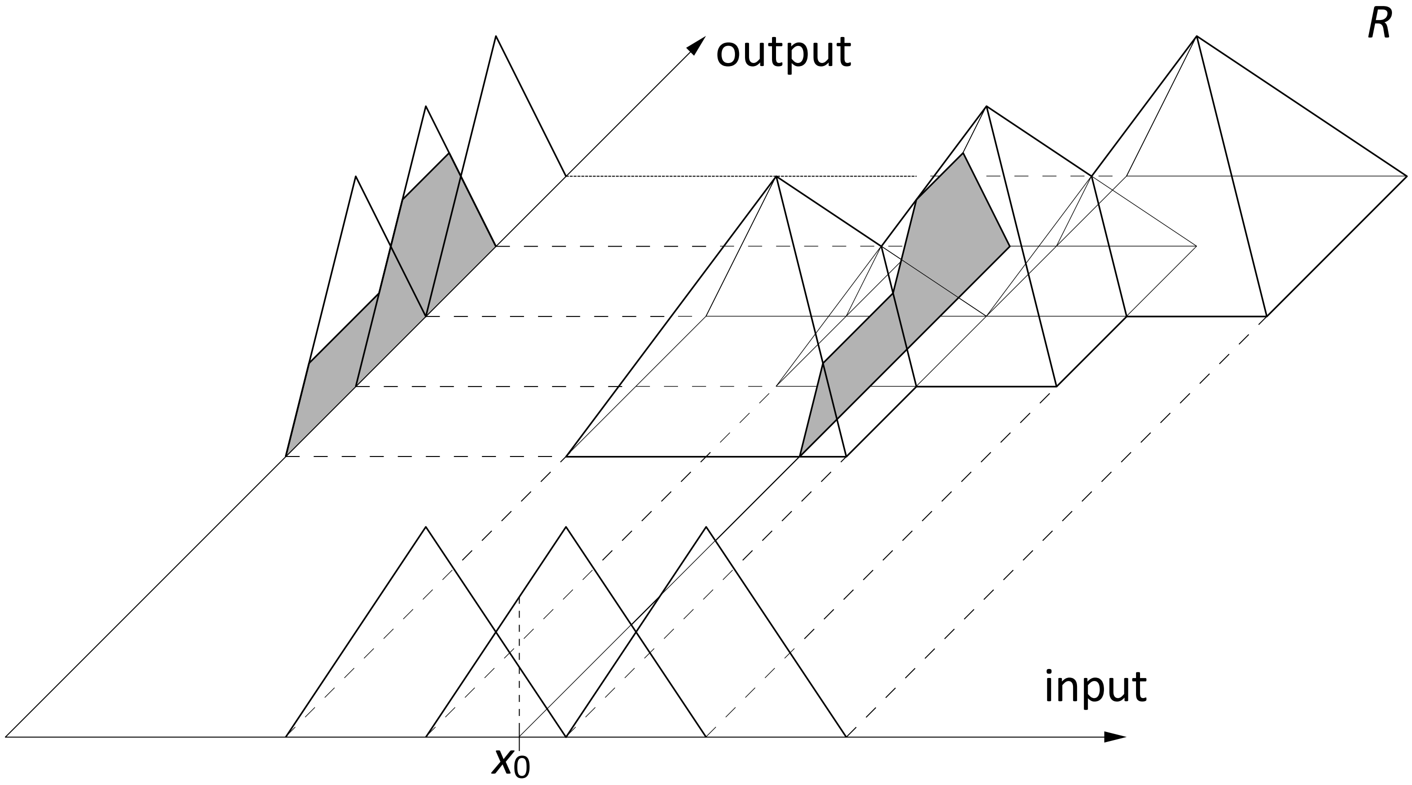
ns → ns', az → az', ps → ps'



$R = \mu_{ns} \times \nu_{ns'} \cup$

$\mu_{az} \times \nu_{az'} \cup \mu_{ps} \times \nu_{ps'}$

Plausible Fuzzy Outputs

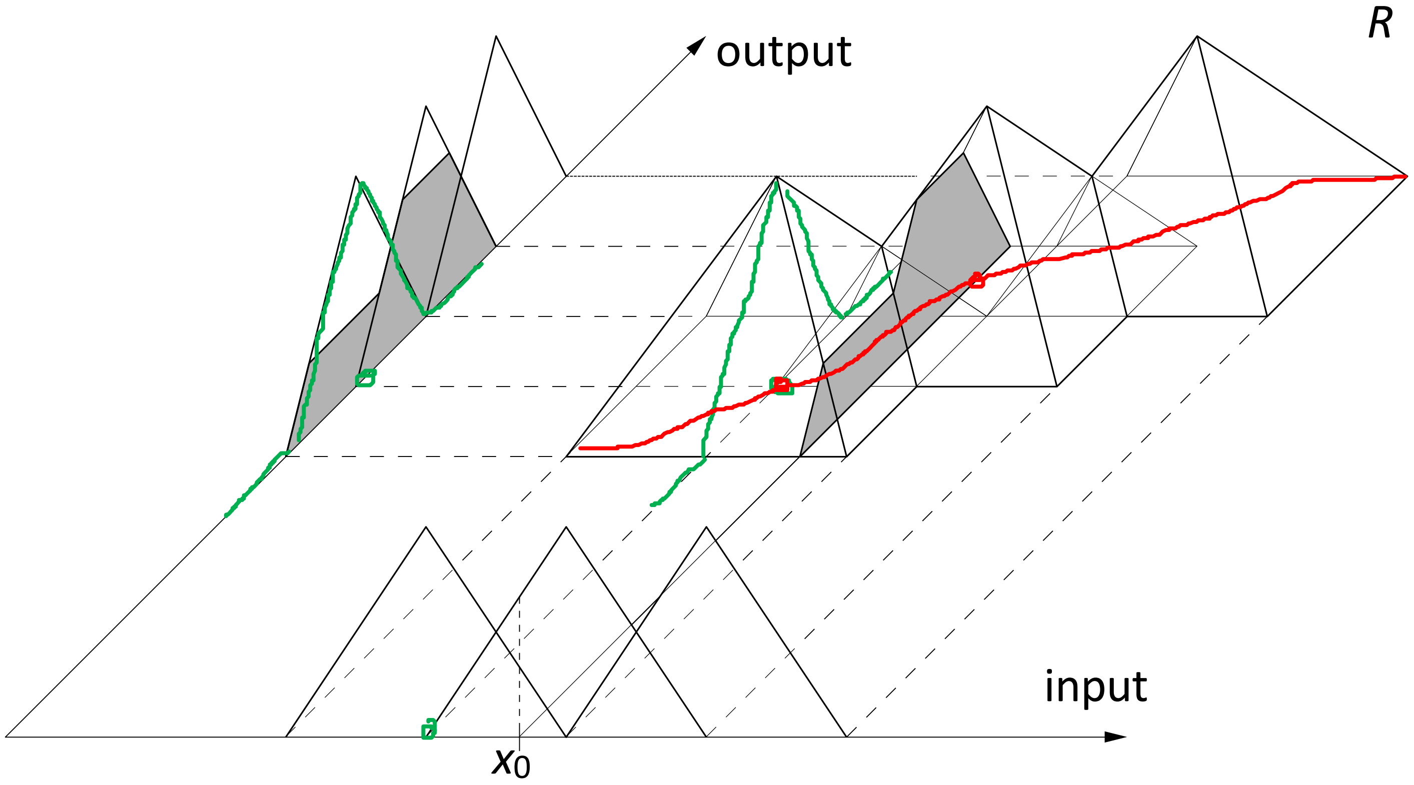


Three fuzzy rules

Every pyramid is specified by a fuzzy rule (Cartesian product).

Input x_0 leads to gray-shaded fuzzy output $\{x_0\} \circ R$.

Definition of a Plausible Input-Output Function

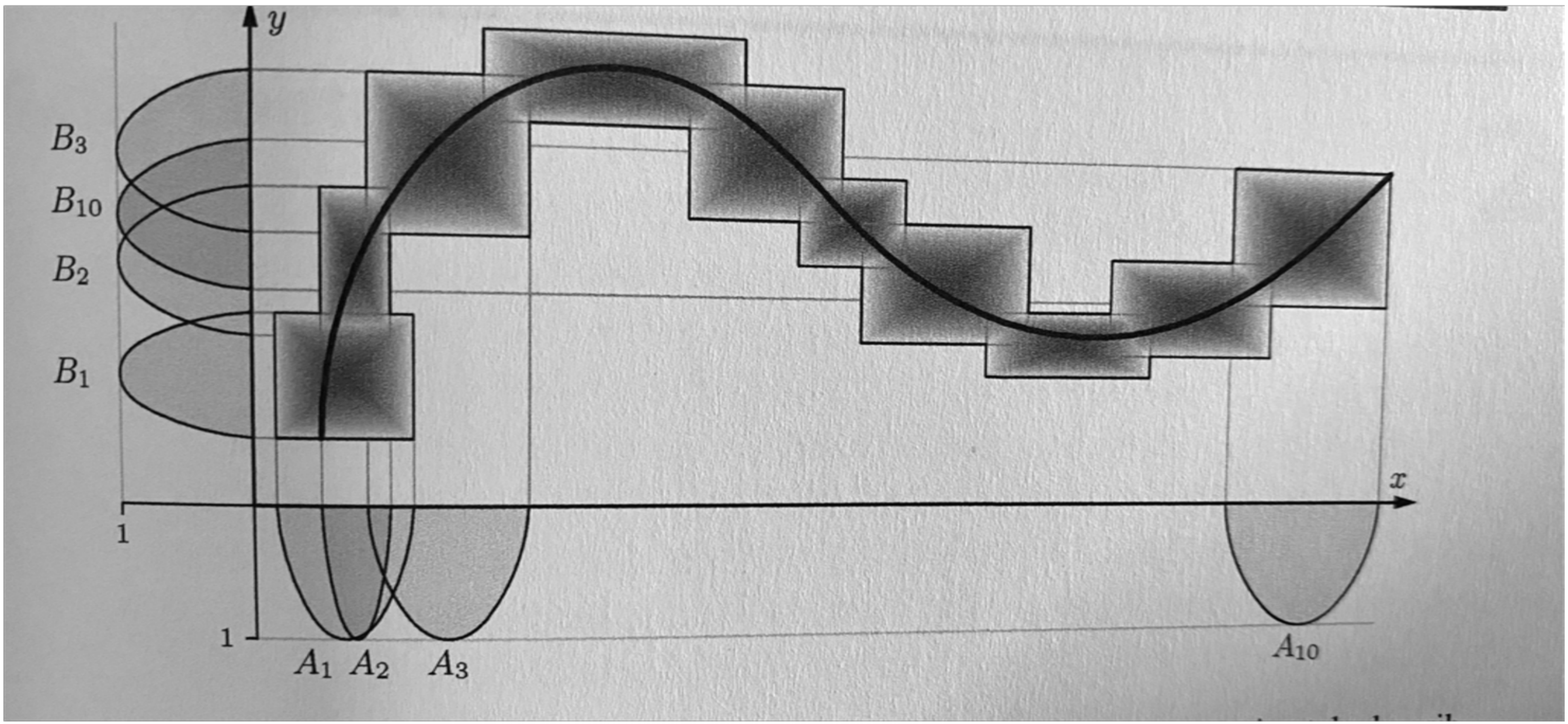


This methods (green) is exactly the same, that for Mamdani Controllers are used before defuzzyfication.

Center of Gravity defuzzyfication gives the control function (red) .

Intuitively it's a kind of interpolation: The function follows the pyramids

Interpolation based on fuzzy information



Interpretation 2

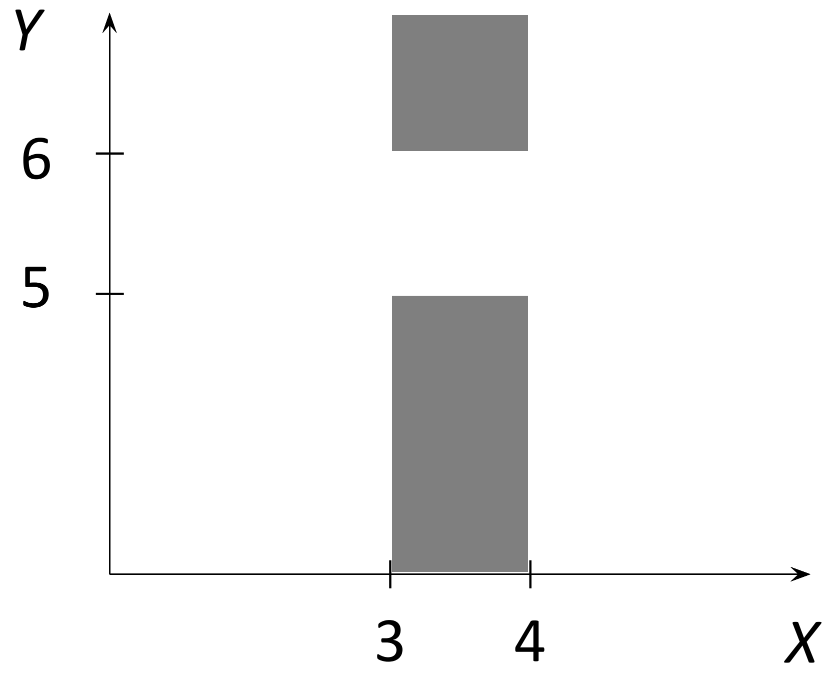
Rules as Logical Constraints

This is a completely different view on a rule

Imprecise rule**: if** *M* = [3, 4] **then** *N* = [5, 6]

Black values are impossible, white ones are allowed.

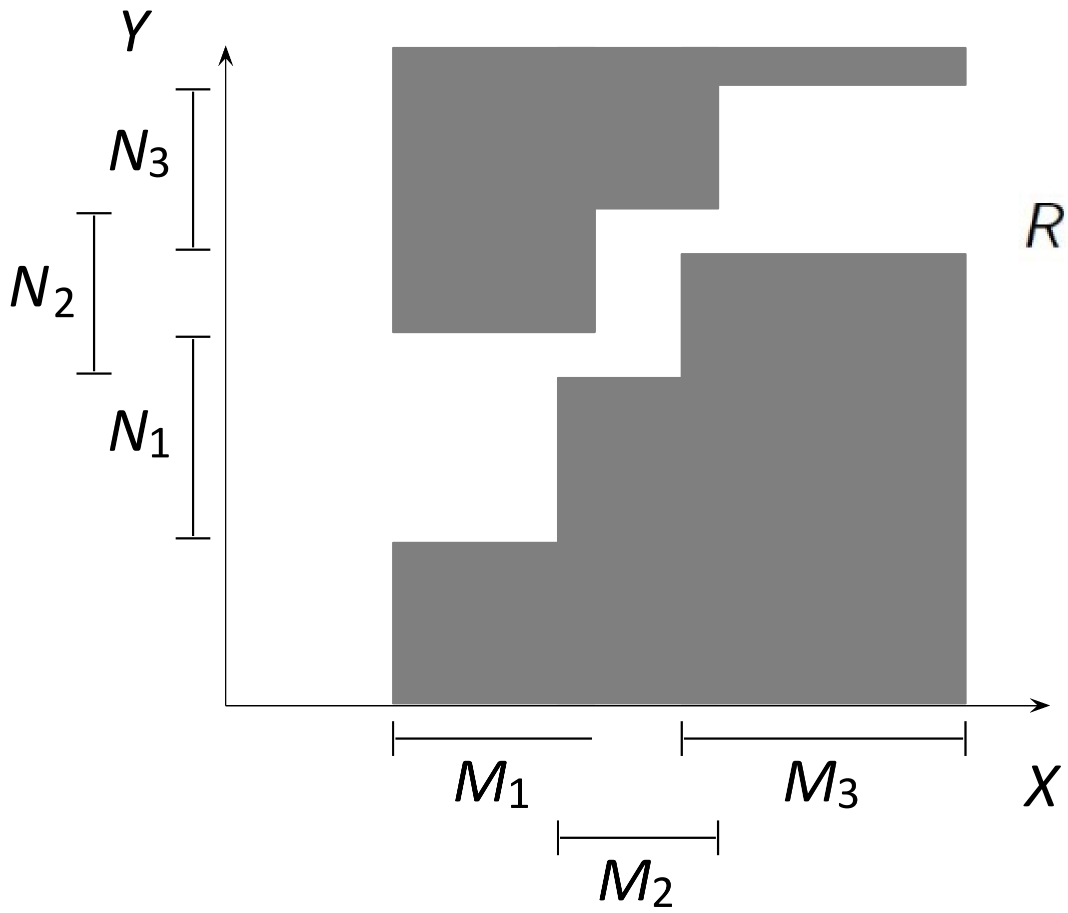
(x,y) allowed if ((x in M and y in N) or (x not in N))



Rules as Constraints

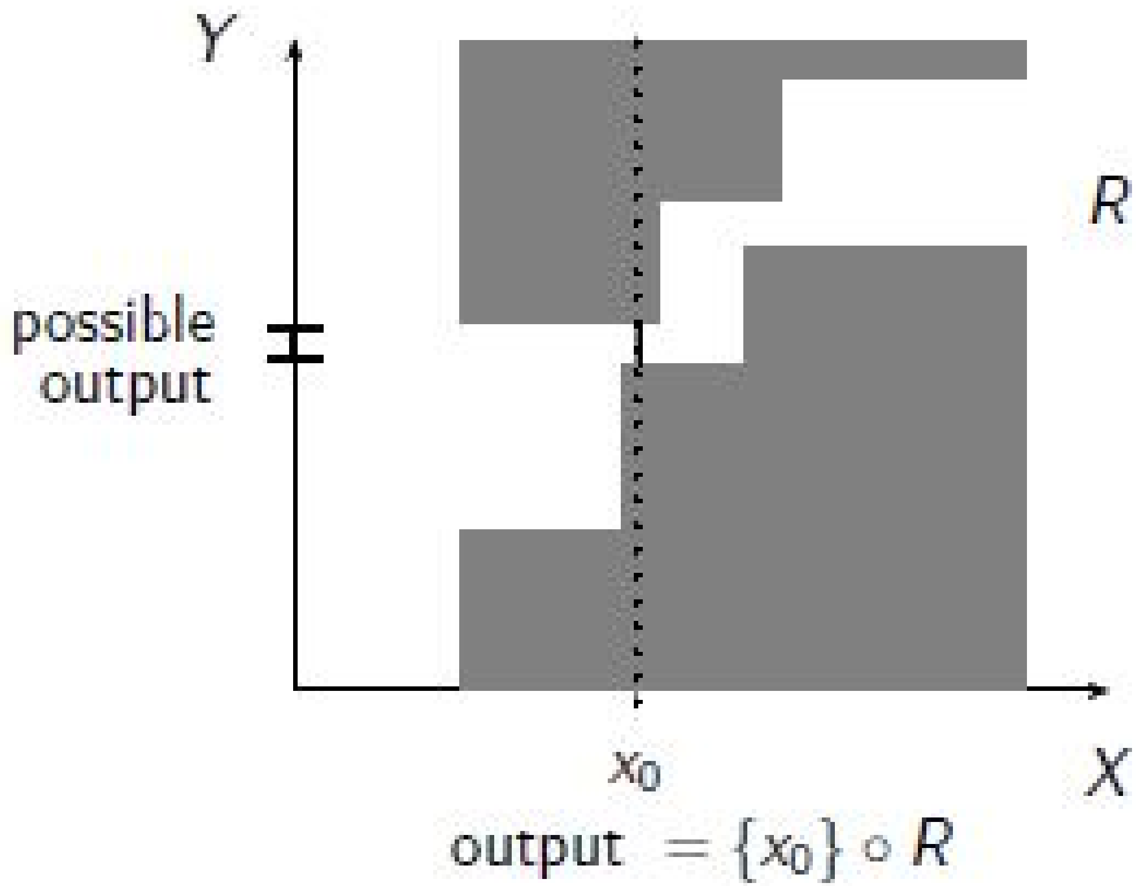
Several imprecise rules: **if** M_1 **then** N_1, **if** M_2 **then** N_2, **if** M_3 **then** N_3

Interpretation: R is the „corridor" (white area) for the function.



$$R = \bigcap_{i=1}^{r} (M_i \times N_i) \cup (M_i^C \times Y)$$

Interpretation 2: Output via composition



Fuzzy Rule as constraints with Zadeh's Logic

Given the Fuzzy Rule:

if μ then η

We interprete the rule as a fuzzy relation R of XxY:

x is in relation R with y if ((x∈μ and y∈η) or not(x ∈ μ)) holds

Using Zadehs logic standard operations we obtain

$$\mu_{R} : X \times Y \to [0,1],$$

$$(x,y) \longrightarrow \max((\min(\mu(x),\eta(y)),1-\mu(x))$$

This corresponds to Zadeh's Implication.

A Fuzzy Rule as Constraint using Gödel‘s Logic

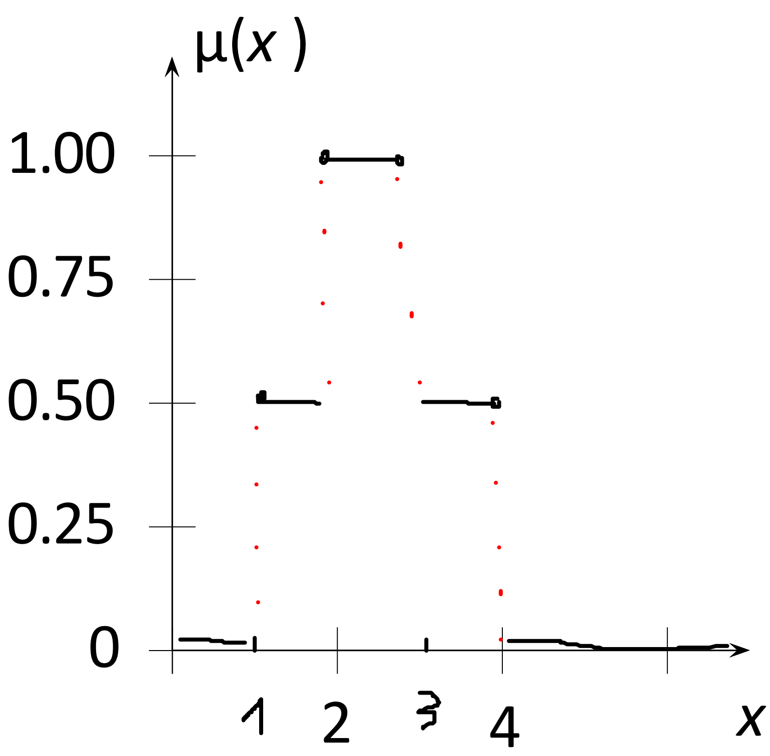
Given the Fuzzy Rule: if μ then ν

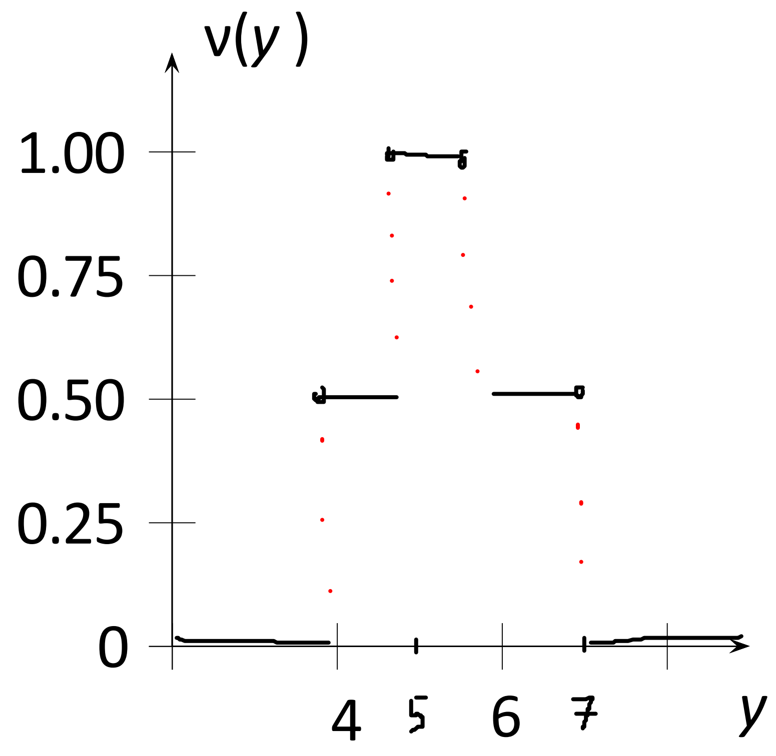
We can use other logics for the interpretation of the rule. The Gödel Logic with its implication ν gives the fuzzy relation R:

$$\mu_R : X \times Y \to [0,1], \quad (x,y) \longrightarrow \begin{cases} 1 & \text{if } \mu(x) \leq \nu(y), \\ \nu(y) & \text{otherwise.} \end{cases}$$

Example

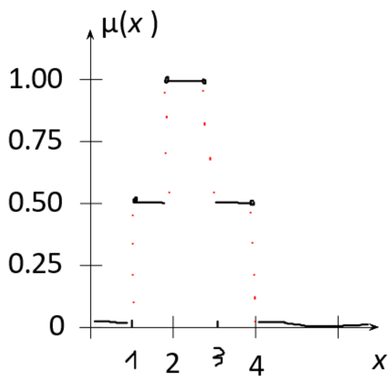
if *x* is approx 2.5 **then** *y* is approx 5.5

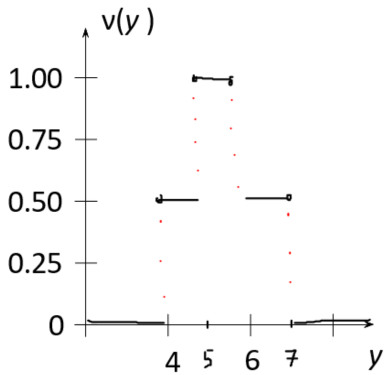


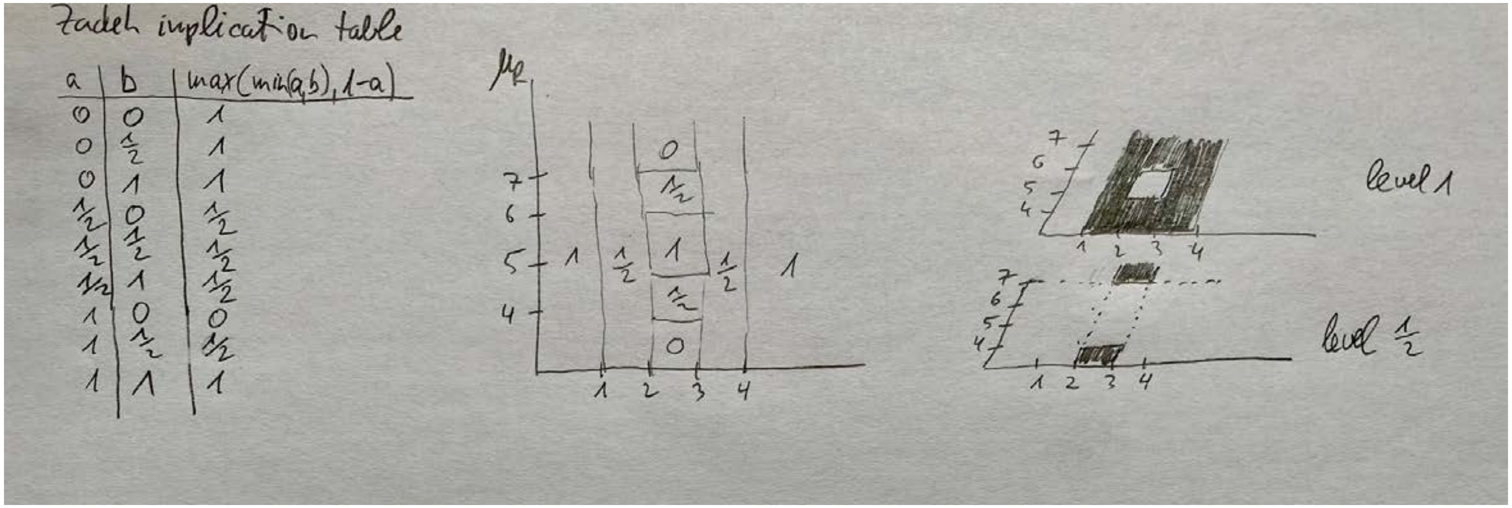


Example: Zadeh‘s Logic

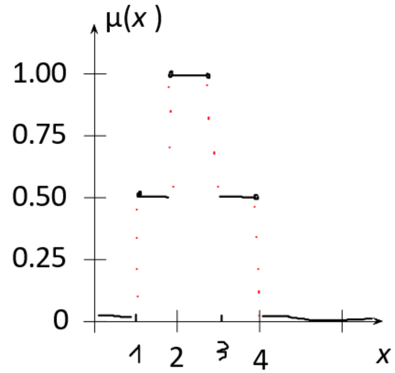
if *x* is approx 2.5 **then** *y* is approx 5.5

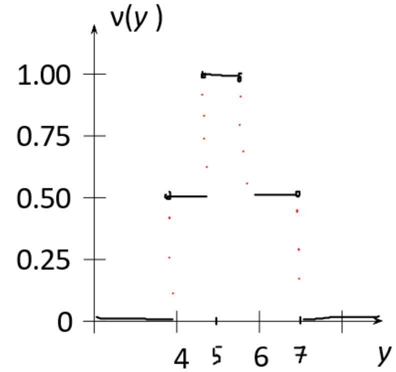


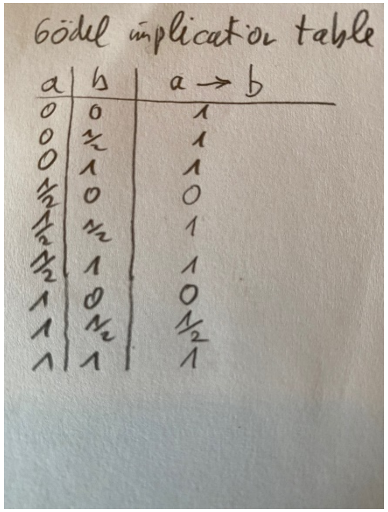




if *x* is approx 2.5 **then** *y* is approx 5.5

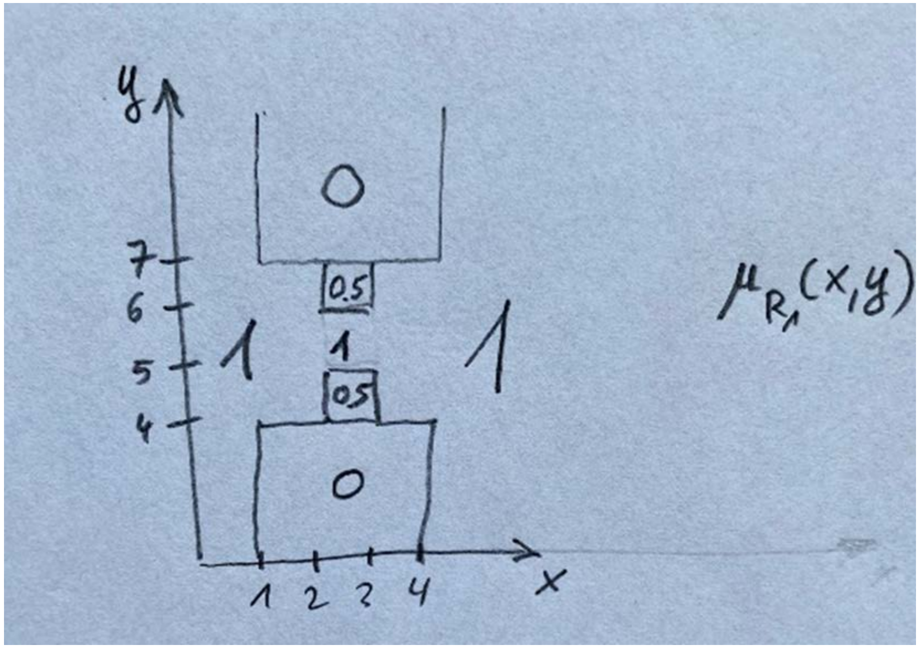






Gödel implication table

a	b	a → b
0	0	1
0	½	1
0	1	1
½	0	0
½	½	1
½	1	1
1	0	0
1	½	½
1	1	1



Example Output via composition

R_1 : **if** $X = \mu_{M_1}$ **then** $Y = \nu_{B_1}$, . . . , R_n : **if** $X = \mu_{M_n}$ **then** $Y = \nu_{B_n}$

The fuzzy relations describing the rules are intersected :
Use the min-Operator for the intersection („and“) and obtain R

$$\mu_R = \min_{1 \le i \le r} \mu_{R_i}$$

For input μ_A and fuzzy relation μ_R the composition gives the output η:

$$\eta(y) = \sup_{x \in X} \min \{ \mu_A(x), \mu_R(x, y) \} .$$

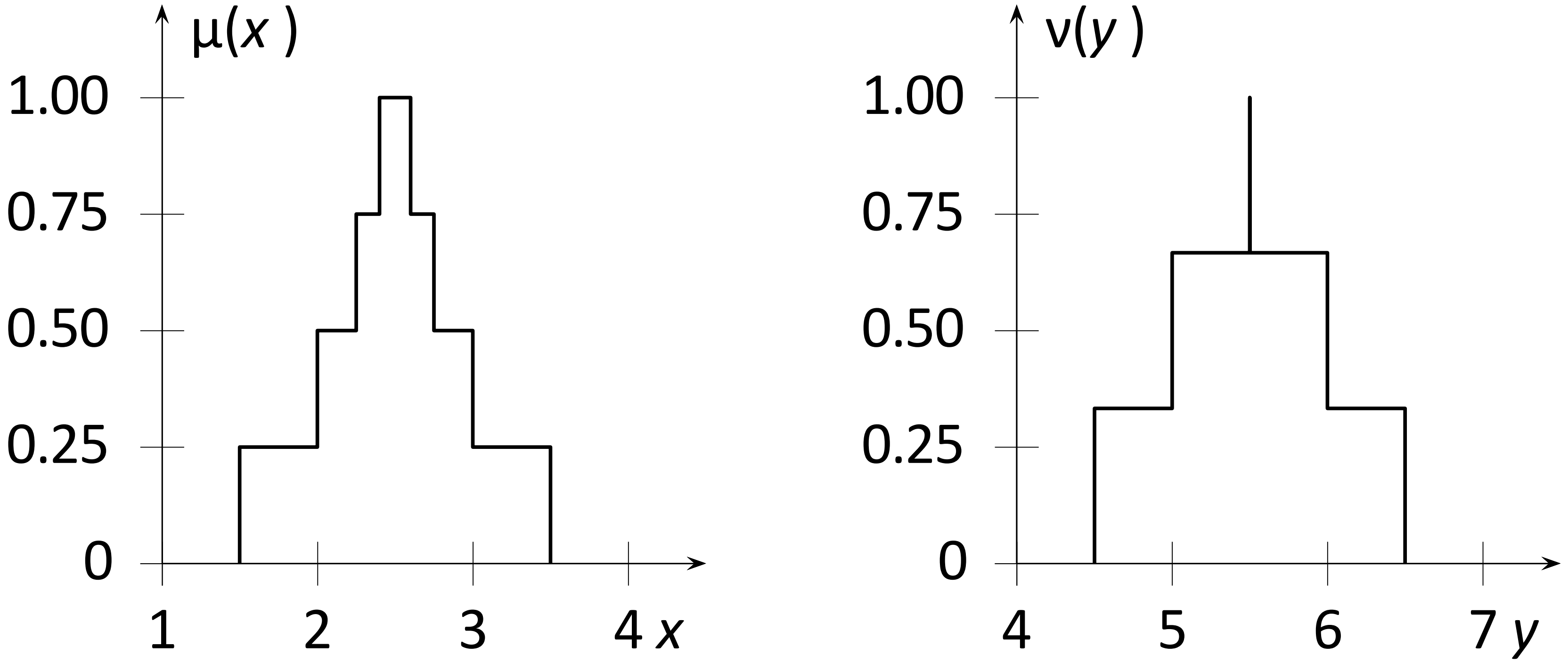
Other operators for „and“, „or“, and „implication“ could be used.
The choice depends on the application.

Interpretation 3

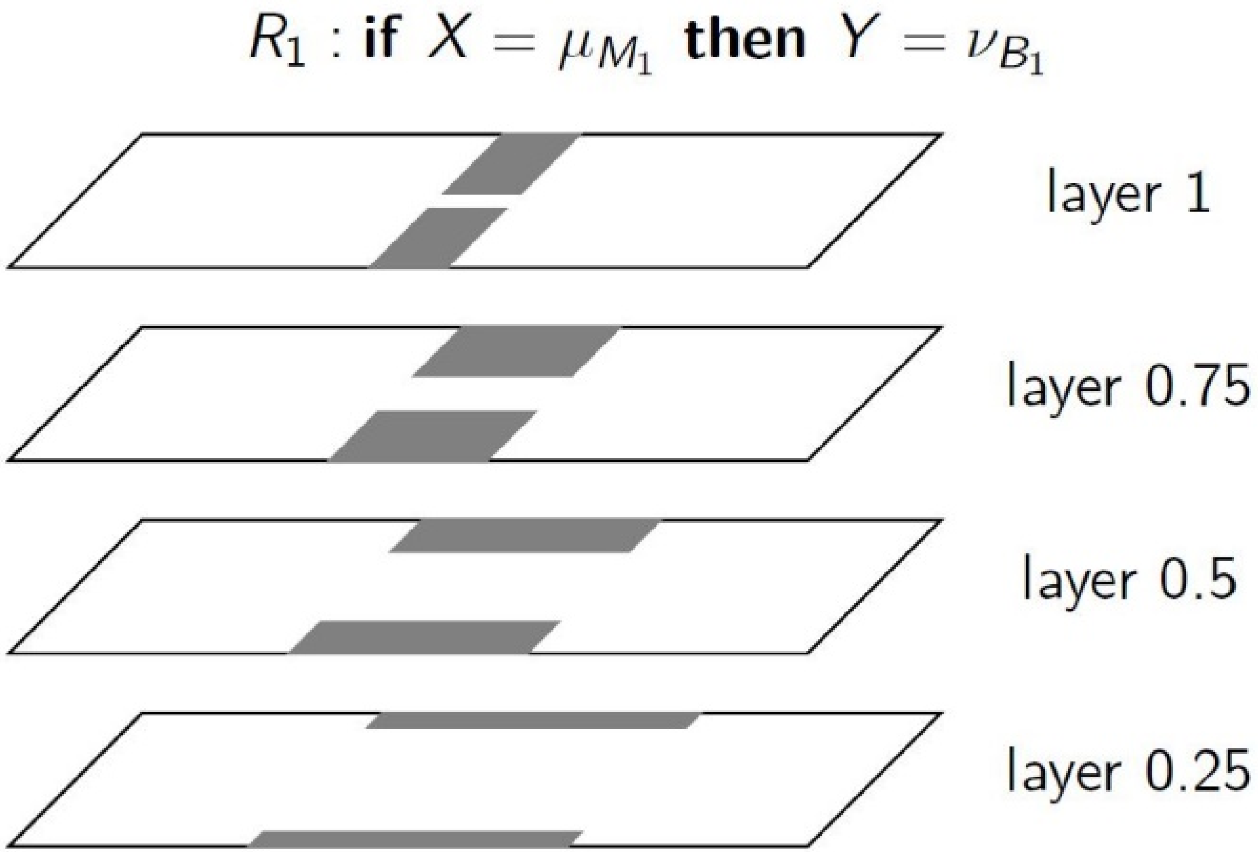
Fuzzy Rule Systems as level-wise constraints using cuts

Example

if *X* is approx. 2.5 **then** *Y* is approx. 5.5

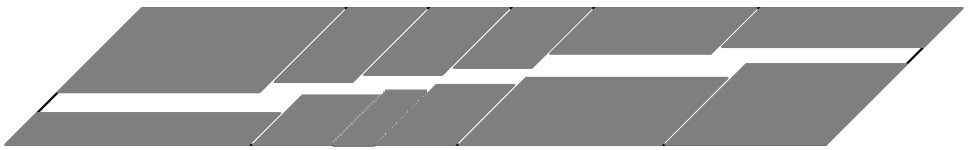


Modeling the constraints levelwise based on cuts



Modeling the constraint levelwise based on cuts

R_1 : **if** $X = \mu_{M_1}$ **then** $Y = \nu_{B_1}$, . . . , R_n : **if** $X = \mu_{M_n}$ **then** $Y = \nu_{B_n}$



layer 1

: :

layer 0.25

$$\mu_R = \min_{1 \leq i \leq r} \mu_{R_i}$$

Interpretation 4

Fuzzy Rule as Fuzzy Relational Equations

Given $\mu_1, \ldots, \mu_r$ of X and $\nu_1, \ldots, \nu_r$ of Y and r rules *if* μ_i *then* ν_i.

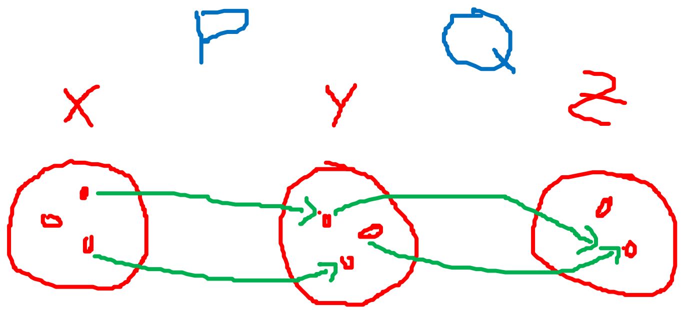
What is a **fuzzy relation** ϱ that fits the rule system?

One solution is to find a relation ϱ such that

$$\forall i \in \{1, \ldots, r\} : \nu_i = \mu_i \circ \varrho,$$

$$\mu \circ \varrho : Y \to [0, 1], \quad y \mapsto \sup_{x \in X} \min\{\mu(x),\ \varrho(x, y)\}.$$

Reminder: Standard Composition



Consider the binary relations $P(X,Y)$, $Q(Y,Z)$ with common set Y.

The **standard composition** of P and Q is defined as

$$(x,z) \in P \circ Q \Longleftrightarrow \exists y \in Y : \{(x,y) \in P \wedge (y,z) \in Q\}.$$

In the fuzzy case this is generalized by

$$[P \circ Q](x,z) = \sup_{y \in Y} \{\min\{P(x,y), Q(y,z)\}\}, x \in X, z \in Z.$$

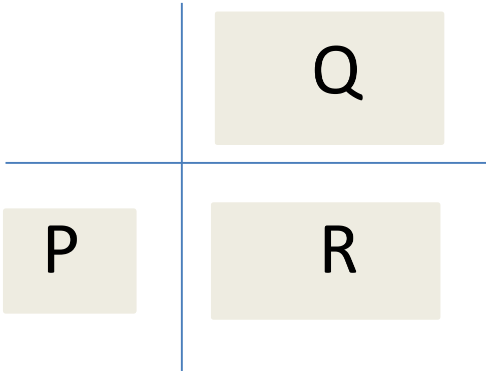
If Y is finite, then sup = max holds.

The standard composition is also called **max-min composition**.

Example

$$P \circ Q = R$$

$$\begin{bmatrix} .3 & .5 & .8 \\ 0 & .7 & 1 \\ .4 & .6 & .5 \end{bmatrix} \circ \begin{bmatrix} .9 & .5 & .7 & .7 \\ .3 & .2 & 0 & .9 \\ 1 & 0 & .5 & .5 \end{bmatrix} = \begin{bmatrix} .8 & .3 & .5 & .5 \\ 1 & .2 & 5 & .7 \\ .5 & .4 & .5 & .5 \end{bmatrix}$$



For instance:

$r_{11} = \max\{\min(p_{11}, q_{11}), \min(p_{12}, q_{21}), \min(p_{13}, q_{31})\}$

$= \max\{\min(.3, .9), \min(.5, .3), \min(.8, 1)\}$

$= .8$

$r_{32} = \max\{\min(p_{31}, q_{12}), \min(p_{32}, q_{22}), \min(p_{33}, q_{32})\}$

$= \max\{\min(.4, .5), \min(.6, .2), \min(.5, 0)\}$

$= .4$

Example

$$\mu_A = (\ .9 \quad 1 \quad .7\ ) \qquad \nu_B = (\ 1 \quad .4 \quad .8 \quad .7\ )$$

$$\varrho_{A \circledS B} = \begin{pmatrix} 1 & .4 & .8 & .7 \\ 1 & .4 & .8 & .7 \\ 1 & .4 & 1 & 1 \end{pmatrix} \qquad \varrho_1 = \begin{pmatrix} 0 & 0 & 0 & .7 \\ 1 & .4 & .8 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{array}{ccc|cccc} & & & 1 & .4 & .8 & .7 \\ & & & 1 & .4 & .8 & .7 \\ & & & 1 & .4 & 1 & 1 \\ \hline .9 & 1 & .7 & 1 & .4 & .8 & .7 \end{array} \qquad \varrho_2 = \begin{pmatrix} 0 & .4 & .8 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & .7 \end{pmatrix}$$

$\varrho_{A \circledS B}$ largest solution, ϱ_1, ϱ_2 are two minimal solutions.

Solution space forms upper semilattice.

Here the maximal solution is the Gödel-Implication.

Gödel Implication

Given the Fuzzy Rule: if μ then ν

The Gödel implication gives the fuzzy relation R:

$$\mu_R : X \times Y \to [0,1], \quad (x,y) \longmapsto \begin{cases} 1 & \text{if } \mu(x) \leq \nu(y), \\ \nu(y) & \text{otherwise.} \end{cases}$$

Solution of a Relational Equation

Theorem

i) Let "if A then B" be a rule with $\mu_A \in \mathcal{F}(X)$ and $\nu_B \in \mathcal{F}(Y)$. Then the relational equation $\nu_B = \mu_A \circ \varrho$ can be solved iff the $\varrho_{A \circledS B}$ is a solution.

$\varrho_{A \circledS B} : X \times Y \to [0,1]$ is defined by

$$(x,y) \mapsto \begin{cases} 1 & \text{if } \mu_A(x) \leq \nu_B(y), \\ \nu_B(y) & \text{otherwise.} \end{cases}$$

ii) If ϱ is a solution, then the set of solutions $R = \{\varrho_S \in \mathcal{F}(X \times Y) \mid \nu_B = \mu_A \circ \varrho_S\}$ has the following property: If $\varrho_{S'}, \varrho_{S''} \in R$, then $\varrho_{S'} \cup \varrho_{S''} \in R$.

iii) If $\varrho_{A \circledS B}$ is a solution, then $\varrho_{A \circledS B}$ is the largest solution w.r.t. $\subseteq$.

Solution of a Set of Relational Equations

Generalization of this result to system of r relational equations:

Theorem

Let $\nu_{B_i} = \mu_{A_i} \circ \varrho$ for $i = 1, \ldots, r$ be a system of relational equations.

i) There is a solution iff $\bigcap_{i=1}^{r} \varrho_{A_i \circledS B_i}$ is a solution.

ii) If $\bigcap_{i=1}^{r} \varrho_{A_i \circledS B_i}$ is a solution, then this solution is the biggest solution w.r.t. $\subseteq$.

Remark: if there is no solution, then the Gödel Implication is good approximation.

If a solution exists, then the space of solutions forms an „upper semi lattice“: The union of two solutions is a solution, the Gödel Implication is the largest solution, the Mamdani methods (used for fuzzy controller) is also a solution, and in general there are several minimal solutions.